PLANNING WITH NON-DECREASING RETURNS TO SCALE

by

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Abstract

The first chapter of this dissertation presents an algorithm in the Malinvaud-Lange tradition. It converges to a global optimum in cases where firms in the economy exhibit non-decreasing returns to scale. A new type of communication between the planning board and the firm is developed: all information is exchanged in the form of quantities; prices and/or marginal rates of substitution are not used.

The second chapter explores the applications of this algorithm to non-convex problems in operations research. Some computational evidence is presented on its speed of convergence.

The third chapter studies a centrally planned economy with both an increasing returns to scale and a decreasing returns to scale sector. It shows how the central planning board can use a price-led procedure to plan the decreasing returns to scale sector and the quantity-led algorithm developed in the first chapter in the increasing returns to scale sector.

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Biographical Note

Jacques Crémer was born in Hyères, France, in 1949. He was admitted to the Ecole Polytechnique in 1968 and graduated in 1970. After serving in the French Navy for a year, he entered the Sloan School of Management at M.I.T., receiving the S.M. in 1973. He escaped the executive suite by registering in the Department of Economics at M.I.T. In July 1974 he married Nancy Sheldon, a Ph.D. candidate in the Department of Urban Studies and Planning at M.I.T. They have a daughter, Marie Louise, born in April 1976.

In the fall of 1975 he worked as an instructor at Northeastern University; from January to August 1976 he was a consultant for the World Bank. During the academic year 1976-1977 he will be doing research at the Laboratorie d'Econométrie de l'Ecole Polytechnique (Paris) and teaching at the Ecole Nationale des Ponts et Chaussées.

CHAPTER I

I. INTRODUCTION

This paper is a contribution to the fast-growing literature on central planning under incomplete information. Since Lange's classic work [4] economists have developed a number of formal procedures by which a Central Planning Board (C.P.B.) can maximize its utility over the production possibilities of an economy with only partial knowledge of the production possibility set of any firm. Typically such procedures are not even well defined in a non-convex environment. When convergence can be proved it is limited to local optima. cf [1] [3]. In this paper we present an algorithm which converges to a global optimum in practically all decomposable environments which make economic sense.

To solve this problem, a new type of communication between the planning board and the firms has to be developed. In the first models of planning procedures the C.P.B. "spoke" in terms of prices while the firms answered with quantities. Later algorithms were developed where the center speaks in terms of quantities and the firms respond with prices -- or, more exactly, marginal rates of substitution. In our procedure all information is exchanged in the form of quantities -- hence the name quantity-quantity.

Hopefully the computational technique presented here can be used to solve a large class of non-convex optimization problems. In this paper we will concentrate our attention on its applications to planning theory. Preliminary investigation of its applicability to operation research problems will be presented in [2].

II. PRELIMINARY DISCUSSION

In Figures 1 and 2 we give a visual representation of the quantity-quantity procedure. We consider the special case of a two-goods economy composed of one firm, whose production possibility set (P.P.S.) is Y. No convexity assumption is made about Y.

The Central Planning Board knows society's utility function u, but does not know Y. The managers of the firm do not know u, and we must therefore devise a method for exchanging information which will enable the C.P.B. to find the best feasible production program for the economy.

At the beginning, the center knows upper bounds w_1 and w_2 on the coordinates y_1^* and y_2^* of y^*, where y^* maximizes u over Y. The point $w = (w_1,w_2)$ is chosen by the center so as to ensure that it is greater or equal than y^*. The set of points smaller or equal than w will be called Y°. Obviously, $y^* \varepsilon Y^\circ$.

The center first asks the firm whether it can produce w. If it can, the problem is solved. If it cannot, the firm must communicate to the center an efficient point, i.e., a point on the boundary of Y, which is strictly smaller than w. In Figure 1a this point is y°.

What does the center learn from the answer of the firm? As y° is efficient it knows that no point of Y is strictly greater than y^{o}. In Figure 1a I have hatched with vertical lines an area which the center can eliminate from future consideration at this stage. The C.P.B. knows that y^{*} must be in Y^{1}, the set of points of Y^{o} which are not greater than y^{o}. From Figure 1a, it is clear that another, more interesting, characterization of Y^{1} is possible: Y^{1} is the set of points smaller or equal to either v^{1} or v^{2}.

At the beginning of the second stage the center maximizes u on Y^{1}. As u will be assumed non-decreasing in all components, it is sufficient to compare its value at v^{1} and v^{2}. Let, for instance, $u(v^{1})$ be greater than $u(v^{2})$. The center asks the firm whether it can produce v^{1}. If the firm cannot, it offers as a compromise y^{1}. The center builds Y^{2}, the set of points belonging to Y^{1}, but not greater than y^{1}. As we can see from Figure 1b, Y^{2} can also be described as the set of the set of points smaller or equal to either v^{2}, v^{3} or v^{4}. The center maximizes u over Y^{2}. In order to do so it is sufficient to choose the greater of $u(v^{2})$, $u(v^{3})$ and $u(v^{4})$. Let us assume it is $u(v^{2})$; y^{2} is chosen strictly smaller than v^{2} and the process is repeated.

In Figure 2 I have represented the state of the process after seven stages. Y^{7} is composed of all points smaller than at least one of z^{1} to z^{8}. As s increases, Y^{s} becomes closer and closer to Y in a neighborhood of y^{*}. The rest of the paper is devoted to a precise formulation of this idea in a more general framework.

III. A MODEL OF A PLANNED ECONOMY

The economy we are studying is composed of m + 1 agents: m firms and the C.P.B. The firms produce and/or use n goods.

Y_k will be the Production Possibility Set (P.P.S.) of the k^{th} firm, $k = 1, \ldots, m$; for all k, Y_k satisfies assumptions[2] A1 to A3:

A1 Y_k is closed

A2 Y_k is bounded from above

A3 If $x' \leqq x$ and $x \varepsilon Y_k$, $x' \varepsilon Y_k$.

A3 is simply an assumption of free disposal. No assumption whatsoever is made about the convexity of the $\{Y_k\}$

Assumptions A1 to A3 satisfy "invariance under mergers": if two firms which satisfy them are combined under the same management the new firm will also satisfy them.

A point x will be said k-feasible if it belongs to Y_k. It will be called k-efficient if it is maximal in the set of k-feasible points, i.e,, if (i) it belongs to Y_k and (ii) $x' > x$ implies $x' \not\in Y_k$.

The center has full authority over the firms. It tries to maximize some function of net output. This utility function, u, satisfies assumptions A4 and A5:

A4 u is non-decreasing in all components

A5 u is upper semi-continuous.

Assumption A4 does not prevent us from considering nuisances; they simply have to be introduced with a negative sign in our model. However, products cannot change from being goods to being nuisances (or vice versa) when their quantity increases. The assumption of upper semi-continuity is more than a cheap generalization: it allows us to model jumps in the planners' satisfaction. Suppose good i represents swimming pool[3] of a certain type; there exists a discontinuity in our utility between 1.99 units of good i and 2 units. This discontinuity can be modelled by an upper semi-continuous function.

The center is also responsible for making sure that the vector of net output belongs to a consumption set X. X satisfies the non-saturation assumption A6 and A7:

A6 If $x \varepsilon X$ and $x' \geqq X$, $x' \varepsilon X$

A7 X is closed

Finally the economy has at its disposition initial stocks of resources -- denoted by the vector ω. The center knows with certainty both ω and X.

The problem, P, facing the C.P.B. can therefore be written:

$$\underline{P} \qquad \begin{aligned} \max \quad & u(x) \\ \text{s.t.} \quad & x \in X \\ & q_k \in Y_k \quad k = 1,\ldots,m \\ & q = \sum_{k=1}^{m} q_k \\ & x \leqq q + \omega \end{aligned}$$

The problem has been set in the framework of national plan construction. Other interpretations are certainly possible. For instance, many of the choices facing large bureaucracies -- whether private or governmental -- could be modelled in this fashion.

To completely specify this problem we must describe the technology which can be used in the planning process. We first assume that the processes of production are so complex that it is impossible for the firms' managers to transmit to the C.P.B. a complete description of the sets Y_k. The assumptions we shall make on the instruments available to the C.P.B. are much less realistic. First, we bestow upon the center a practically unlimited computational capability so that we need not concern ourselves with the size of the computations it will have to perform. Furthermore, we assume that communications between the C.P.B. and the firms are at the same time very quick and very cheap.

IV. THE QUANTITY-QUANTITY ALGORITHM

Before beginning the planning process the center knows all the data necessary to solve problem P except the sets Y_k. It is therefore natural to proceed by approximating those sets.

At stage s, the C.P.B. builds an estimate Y_k^s of the P.P.S. of firm k, $k = 1,\ldots,m$. It replaces Y_k by Y_k^s in P and solves this "problem of approximation" which we will name P^s:

$$
\underline{P^s} \qquad
\begin{array}{ll}
\text{Max} & u(x) \\
\text{s.t.} & x \varepsilon X \\
 & q_k \varepsilon Y_k^s \quad k = 1,\ldots,m \\
 & q = \sum_{k=1}^{m} q_k \\
 & x \leqq q + \omega
\end{array}
$$

The solution of P^s provides the C.P.B. with a tentative plan $(q_1^s,\ldots,q_k^s,\ldots,q_m^s)$ which is proposed to the firms. In the quantity-quantity procedure this plan will, in general, be infeasible. Therefore the firms offer a compromise, a feasible point not too far, in some specific way to be precised later, from the center's proposal. From those answers new approximate sets Y_k^{s+1} can be built and the process is repeated until the plan $\{q_k^s\}$ converges to a solution of P.

We will use the rest of this Section to make precise the description we have just sketched. We will first discuss the choice of the set Y_k^o, then the construction of the set Y_k^{s+1} from Y_k^s and finally some properties of the sequence of sets $\{Y_k^s\}$.

We assume that, at the outset, the center knows something about one of the solutions $(y_1^*,\ldots,y_m^*)$ of P: for all k it knows a w_k such that y_k^* is smaller or equal to w_k. As w_k does not need to satisfy any other requirement this seems a very weak assumption. In real life w_k would be built by, say, doubling all outputs required of firm k in the last planning period and dividing by two all the inputs it was allocated. Note that $(w_1,\ldots,w_m)$ is not infeasible because it misallocates responsibilities between firms but because it requires too much of every one of them.

The center builds Y_k^o from w_k: Y_k^o is the set of points smaller or equal to w_k. Formally: $Y_k^o = \{x \mid x \leqq w_k\}$. Obviously y_k^* belongs to Y_k^o for all k.

Let $q^s = (q_1^s,\ldots,q_m^s)$ be a solution of P^s. The center asks firm k whether it can produce q_k^s. Assume first that the answer is negative. The firm proposes as a compromise a point y_k^s, k-efficient and strictly smaller than q_k^s.

No point strictly greater than y_k^s belongs to Y_k, by the definition of k-efficiency. Y_k^{s+1}, our new approximation, is therefore built by subtracting from Y_k^s all the points strictly greater than y_k^s. Formally:

$$Y_k^{s+1} = \{x \mid x \varepsilon Y_k^s,\ x \not> y_k^s\} \qquad (1)$$

If q_k^s is feasible we have $y_k^s = q_k^s$ and define $Y_k^{s+1} = Y_k^s$.

The process is then repeated starting with problem P^{s+1}. Obviously we have:

$$Y_k^o \supseteq Y_k^1 \supseteq \cdots \supseteq Y_k^s \supseteq \cdots \qquad (2)$$

Since no point of Y_k is strictly greater than y_k^s, all points of Y_k which belong to Y_k^s will belong to Y_k^{s+1}:

$$Y_k^{s+1} \cap Y_k \supseteq Y_k^s \cap Y_k$$

and by (2): $$Y_k^{s+1} \cap Y_k \subseteq Y_k^s \cap Y_k$$

Therefore $$Y_k^o \cap Y_k = Y_k^s \cap Y_k$$

and $$y_k^* \varepsilon Y_k^s \quad \text{for all } s \text{ and all } k \qquad (3)$$

Obviously Y_k^o satisfies A3, the assumption of free disposal. We will show by induction that so does Y_k^s, for all s. Assume Y_k^{s-1} satisfies A3. Let $x \varepsilon Y_k^s$ and $x' \leqq x$. By (2) x belongs to Y_k^{s-1} and therefore so does x'. But $x \varepsilon Y_k^s$ also implies $x \not> y_k^{s-1}$; there exists an i such that $x_i' \leqq x_i \leqq y_{ik}^s$, the i^{th} coordinate of y_k^s. This implies $x_i' \not> y_{ik}^s$; hence $x' \varepsilon Y_k^s$.

We have proved:

$$x \varepsilon Y_k^s,\ x' \leqq x \Rightarrow x' \varepsilon Y_k^s \qquad (4)$$

V. APPROXIMATING THE PRODUCTION POSSIBILITY SETS OF THE FIRMS

In this Section we will show that for all s and all k there exist finite sets D_k^s such that Y_k^s is composed of all points smaller than or equal to at least one point of D_k^s. Formally:

$$x \varepsilon Y_k^s \Longleftrightarrow \exists v \varepsilon D_k^s \text{ such that } x \leqq v \qquad (5)$$

D_k^s will be said to generate Y_k^s

This result will give us a simple rule for solving problem P^s: we merely compare the value of u at a finite number of points. Indeed, let $(\bar{q}_1^s, \ldots, \bar{q}_m^s)$ be a solution of P^s. By (5), for all k, there exists a $v_k^s \varepsilon D_k^s$ such that $\bar{q}_k^s \leqq v_k^s$. Therefore, as u is non-decreasing, we will have $u(\sum_{k=1}^m \bar{q}_k^s + w) \leqq u(\sum_{k=1}^m v_k^s + w)$; $(v_1^s, \ldots, v_m^s)$ is a solution of P^s. To find a solution of P^s, it is therefore sufficient to solve the finite problem $\overline{P}^s$:

$$\begin{array}{ll} \text{Max} & u(x) \\ \text{s.t.} & x \varepsilon X \end{array}$$

$$q^k \varepsilon D_k^s \quad k = 1,\dots,m$$

$$\bar{P}^s \qquad q = \sum_{k=1}^{m} q^k$$

$$x \leqq q + \omega$$

Obviously D_k^o exists: Y_k^o has been defined as the set of points smaller than or equal to w_k. Therefore, the set $\{w_k\}$, whose only element is w_k, generates Y_k^o.

Let us now assume that D_k^s exists. We have solved P^s by the method indicated above, so that q_k^s belongs to D_k^s for all k. y_k^s will be chosen strictly smaller than q_k^s. There might actually be other points of D_k^s strictly greater than y_k^s (Figure 1 can be misleading on this point: in two dimensions only one such point exists, but this property does not hold in general).

In any case, let us consider a $\bar{v}$ belonging to D_k^s and strictly greater than Y_k^{s+1}. Obviously $\bar{v}$ does not belong to Y_k^{s+1}. Let ${}_i\bar{v}, i = 1,\dots,n,$ be the point defined by (6) and (7).

$${}_i\bar{v}_i = y_{ik}^s \qquad (6)$$

$${}_i\bar{v}_j = \bar{v}_j \text{ for all } i \text{ and all } j \neq i \qquad (7)$$

As ${}_i\bar{v}$ is smaller than $\bar{v}$ it belongs to Y_k^s by (4). By

(6) it is not strictly greater than y_k^s and therefore by (1) it belongs to Y_k^{s+1}.

We can go further and show that the set of points smaller than $\bar{v}$ and belonging to Y_k^{s+1} is generated by the ${}_i\bar{v}$, $i = 1,\ldots,n$.

Consider first an x such that $x \leqq {}_i\bar{v}$. As ${}_i\bar{v}$ belongs to Y_k^{s+1}, by (1) x belongs to Y_k^{s+1}.

Assume now that x belongs to Y_k^{s+1}, and is smaller than $\bar{v}$. As x belongs to Y_k^{s+1} it cannot be strictly greater than y_k^s, and therefore $x_i \leqq y_{ik}^s = {}_i\bar{v}_i$ for some i. Furthermore $x \leqq \bar{v}$ implies $x_j \leqq \bar{v}_j = {}_i\bar{v}_j$ for all $j \neq i$. We have proved that x is smaller than or equal to ${}_i\bar{v}$.

Consider now a point v' belonging to D_k^s and not strictly greater than y_k^s. Obviously $v' \varepsilon Y_k^{s+1}$. By (4) all $x \leqq v'$ also belong to Y_k^{s+1}.

As Y_k^{s+1} is a subset of Y_k^s all points of Y_k^{s+1} are smaller than some $v \varepsilon D_k^s$. From the preceding discussion it follows that Y_k^{s+1} is generated by the set D_k^{s+1} of all v' and all ${}_i\bar{v}^s$. Formally:

$$D_k^{s+1} = \{v \mid v \varepsilon D_k^s \text{ and } v \not> y_k^s \text{ or } v = {}_i\bar{v} \text{ for some } \bar{v} \varepsilon D_k^s,\ \bar{v} > y_k^s\} \qquad (8)$$

A simple numerical example might help clarify the economics of the matter. Firm 1 is producing econometric studies (good 1) using brains (good 2) and computers (good

3) as inputs. From previous experience the C.P.B. knows that the optimum plan will not ask the firm to produce more than 100 studies, and will not assign it less than 10 brains and 3 computers. Remembering to count inputs negatively we can write $w_1 = (100,-10,-3)$. The firm cannot produce 100 studies with such a small input but offers to produce $y^o = (50,-20,-4)$. The center can now build D_1^1; it is composed of three points ${}_1v = (50,-10,-3)$, ${}_2v = (100,-20,-3)$ and ${}_3v = (100,-10,-4)$. The center knows, for instance, that with less than 4 computers and less than 20 brains the center cannot produce more than 50 studies.

We have mentioned above the possibility for D_k^s to have more than one point greater than y_k^s. Figure 3 allows us to show this on a simple example.[4] Assume that $q_1^1 = {}_1v$. Nothing in the structure of the problem rules out $y_1^1 = (40,-15,-6)$. At the same time we would have ${}_1v > y_1^1$ and ${}_3v > y_1^1$.

VI. CONVERGENCE

Before proving that the sequence $\{q_k^s\}$ converges to a solution of the problem P we must impose a slightly more stringent condition on the choice of y_k^s by the firms. In Figure 1b we see that no information would have been gained by the C.P.B. if the firm had proposed h instead of y_k^s as a compromise. This is the reason why we required y_k^s to be strictly smaller than q_k^s. However, this requirement is not enough for the procedure to converge. We must prevent the firm from giving less and less information as time goes by; i.e., no component of the vector $(q_k^s - y_k^s)$ must be allowed to converge to zero faster than the others. We will ensure this by a simple additional requirement on y_k^s: for all s and all k such that $q_k^s \neq y_k^s$, $q_k^s - y_k^s$ must belong to a close cone C_k which is a subset of the positive quadrant. Formally let R^{n+} be the positive quadrant, $R^{n+} \doteq \{x | x \geqq 0\}$. C_k is closed and belongs to $\overset{\circ}{R}{}^{n+} = \{x | x > 0\}$.

This additional requirement on the answers of the firms is normatively undesirable; it would be much easier to ask the firms questions of the type: "How much output could you produce with those inputs?" Our need for this additional requirement points out the fact that the easy questions might be those which produce the less useful information.

We should now check that the procedure is what Malinvaud [5] calls "well-defined", i.e., that every step can actually be carried out. The only aspect which raises any problem is the existence of an answer to the question asked by the center.[5]

At stage s the center has asked firm k whether it can produce a k-efficient point smaller than q_k^s. If q_k^s is feasible we are done. In the following we will assume it is not. Let e be a vector of C_k -- e is strictly positive. We will prove that there exists a $\lambda^m > 0$ such that $\zeta^m = q_k^s - \lambda^m e$ belongs to Y_k and is k-efficient; ζ^m will be an answer to the question asked by the center as $q_k^s - \zeta^m = \lambda^m e$ obviously belongs to C_k.

It is clear that for any q_k^s, $q_k^s - \bar{\lambda} e$ will be smaller than y_k^* for some finite $\bar{\lambda}$ large enough; $q_k^s - \bar{\lambda} e$ will then belong to Y_k by the free disposal assumption. The set of $\lambda : \{q_k^s - \lambda e \varepsilon Y_k\}$ is therefore non-empty. As Y_k is closed, it is also closed. Finally q_k^s does not belong to Y_k so that λ has a lower bound, 0. From all this we know there exists a λ^m which is a minimum of the set of feasible λ. Is $\zeta^m = q_k^s - \lambda^m e$ k-efficient? Assume it was not and consider $\zeta > \zeta^m$, $\zeta \varepsilon Y_k$. There would exist a $\lambda' > 0$ such that $\zeta^m + \lambda' e$ is smaller than ζ and therefore belongs to Y_k. But then $(\lambda^m - \lambda')$ would belong to $\{\lambda \mid q_k^s - \lambda e \varepsilon Y_k\}$ which is not possible if λ^m is to be the lower bound of this set. This is the contradiction we were looking for.

We need one last preliminary remark. At this point we cannot guarantee that the sequence $\{q_k^s\}$ will converge. The reason is illustrated by Figure 4a. In this example the economy has only one firm; we are trying to maximize the output of good 2. The sequence q_1^s does not converge. There are a number of ways of ruling out such behavior. One of them is illustrated by Figure 4b. We assume that, for all k, the set W_k of points which are smaller than or equal to w_k and do not belong to the interior of Y_k is bounded. This assumption does not seem very restrictive economically.

For all s, q_k^s belongs to W_k. Therefore $\{q_k^s\}$ will have at least one accumulation point. Furthermore, there must exist subsequences of $\{q_k^s\}$, k=1,...,m, which converge simultaneously to points $\{\bar{q}_k\}$. Let $\bar{q} = \sum_{i=1}^{m} \bar{q}_k$.

Our proof of convergence will be divided into three steps. First we will show that, for all k, $\bar{q}_k$ belongs to Y_k. Then we will show that the point $(\bar{q} + \omega)$ belongs to X. Finally, we will prove that it is a solution of P. Of these three steps only the first is not trivial.

<u>Proposition 1</u>: $\bar{q}_k$ belongs to Y_k.

Some of the definitions used in the proof of this proposition are illustrated in Figure 5.

Proposition 1 is obvious if the algorithm converges in a finite number of steps. Suppose it does not and that $\overline{q}_k$ does not belong to Y_k. As Y_k is closed there exists a compact set F such that $\overline{q}_k$ belongs to the interior of F, and the intersection of F and Y_k is empty.

For all $x \varepsilon F$ define $C_k(x)$ in the following manner: y belongs to $C_k(x)$ if and only if it is k-efficient and (x-y) belongs to C_k. $C_k(x)$ is closed because both C_k and Y_k are and it is bounded as it is a subset of the bounded set W_k. Therefore it is compact.

Let x be a vector of F, y in $C_k(x)$ and z a vector of R^n which does not belong to R^{n+}. We will define d(x-y,z) as the distance between x-y and z and $\varepsilon(x,y)$ as the lower bound of d(x-y,z) for $z \not\varepsilon R^{n+}$. As x-y belongs to C_k which is closed and interior to R^{n+} ε (x,y) is strictly positive.

$\varepsilon(x)$ will be the lower bound of $\varepsilon(x,y)$ for y belonging to $C_k(x)$. By compactness of $C_k(x)$, $\varepsilon(x)$ is a minimum and therefore strictly positive for all x.

Finally let us call ε the lower bound of $\varepsilon(x)$ on F. As F is compact we can apply the same reasoning as in the preceding paragraph to show that ε is strictly positive.

We can summarize this discussion:

$$x \varepsilon F,\ y \varepsilon C_k(x),\ z \not\varepsilon R^{n+} \quad d(x-y,z) \geq \varepsilon > 0 \qquad (8)$$

Consider a q_k^s belonging to the interior of F such that $d(\overline{q}_k - q_k^s) \leq \varepsilon/2$; such a q_k^s necessarily exists. Let

$z \varepsilon R^{n+}$. We have:

$$(\bar{q}_k - y_k^s) - z = (\bar{q}_k - q_k^s) + (q_k^s - y_k^s) - z$$

This implies:

$$d\ (\bar{q}_k - y_k^s,\ z) \geq |\ d\ (q_k^s - y_k^s,\ z) - d\ (\bar{q}_k,\ q_k^s)\ |$$

As $y_k^s \varepsilon C_k(q_k^s)$ and by (8):

$$d\ (\bar{q}_k - y_k^s,\ z) \geq \varepsilon - \varepsilon/2 = \varepsilon/2 \qquad (9)$$

Equation (9) holds for every $z \notin R^{n+}$; therefore $(\bar{q}_k - y_k^s)$ belongs to the interior of R^{n+}; $\bar{q}_k$ is strictly greater than y_k^s and does not belong to Y_k^{s+1}. By (2) q_k^{s+m}, $m > 0$, belongs to Y_k^{s+1} which is a closed set; therefore $\bar{q}_k$ cannot be the limit of $\{q_k^s\}$ which is the contradiction we were looking for.

<u>Proposition 2</u>: $\bar{q} = \sum_{k=1}^{m} \bar{q}_k$ is feasible.

For all s we have $\sum_{k=1}^{m} q_k^s + \omega \varepsilon X$; as X is closed $\bar{q} + w$ will also belong to X. Furthermore, by Proposition 1, $\bar{q}_k \varepsilon Y_k$ for all k and therefore $\bar{q}$ is feasible.

<u>Proposition 3</u>: $\overline{q}$ is a solution of P.

By (3) and (5) for every s and every k there exists a point $t_k^s \varepsilon D_k^s$ such that y_k^* is smaller than or equal to t_k^s. Let $t^s = \sum_{k=1}^{m} t_k^s$ and $y^* = \sum_{k=1}^{m} y_k^*$. We have $y^* + \omega \leqq t^s + \omega$, and therefore, by the non-satiation hypothesis, $t^s + \omega$ belongs to X. As $q^s = \sum_{k=1}^{m} q_k^s$ is a solution of P^s we have:

$$u(\overline{q} + \omega) \leq u(t^s + \omega) \leq u(q^s + \omega)$$

Therefore, by upper and semi-continuity of u:

$$u(\overline{q} + \omega) = u[\lim(q^s + \omega)]$$

$$\geq \lim\sup u(q^s + \omega)$$

$$\geq u(y^* + \omega)$$

As y^* is a solution of P and $\overline{q}$ is feasible we have $u(\overline{q} + \omega) = u(y^* + \omega)$ and $\overline{q}$ is a solution of P.

VII. CONCLUDING REMARKS

Of course in all applications we would have to stop the procedure before reaching an optimum. What should the planner do in this case? The answer to this question is easy for procedures such as Malinvaud's [5] where the proposals of the firms form at every stage a feasible program. It is sufficient to choose the $y^s = \sum_{k=1}^{m} y_k^s$ with the highest utility. In production target procedures, it is not possible to guarantee that a feasible sequence $\{\sum_{k=1}^{m} y_k^s\}$ converges to the optimum: $\sum_{k=1}^{m} y_k^s + \omega$ might very well not belong to the set X.

However, in practical situations we can expect that for s big enough there will exist a set of integer indices $(s_1,\ldots,s_m)$, $s_i \leq s$, such that $\sum_{k=1}^{m} y_k^{s_k}$ belongs to X. The center can choose as final plan one of those $\sum_{k=1}^{m} y_k^{s_k}$ which gives the highest utility.

Up to this point we have assumed that all the quantitities of all goods can vary continuously. With only minor changes it is possible to handle situations where goods come only in discrete quantities[6] (e.g., the number of lanes of a highway). In this case a firm might have to propose at some states several points y_k^s. This is illustrated in Figure 6. The economy is similar to the one discussed in Section II, but good 1 came only in discrete quantity. I have represented the situation after 5 stages. We see that no feasible point is strictly smaller than q_1^s; the form will have to offer two points y_1^s; a and b.

It is clear that the quantity-quantity algorithm is not effective in the case where the production possibility sets of the firms are known to be convex as it does not use the information available on the special structures of such economies.[7]

In most economies, however, there are a few big firms -- presumably acting under increasing returns to scale (I.R.S.) -- and a large number of small firms acting under constant or decreasing returns to scale (D.R.S.). For such economies it is possible to design composite planning algorithms. In the I.R.S. sector the center uses the quantity-quantity algorithm while a specialized "global" algorithm is used in the D.R.S. sector. Two of those algorithms are discussed in [2].

1) This paper is part of a Ph.D. dissertation at M.I.T. I wish to thank Vince Crawford, Yves Balcer and especially Professor Martin Weitzman, my thesis advisor.

2) Notation: Let a, b ε R^n. $a \leqq b$ (a is smaller than or equal to b) means $a_i \leq b_i$, $\forall_i$. $a \leq b$ (a is smaller than b) means $a_i \leq b_i$, $\forall_i$, $a \neq b$. $a < b$ (a is strictly smaller than b) means $a_i < b_i$, $\forall_i$.

3) We assume that 1.99 swimming pools (whatever the precise meaning) is better than 1, for instance, because it can be finished during the next planning period. If not, it is easier to model swimming pools as discrete variables (see Section VI).

4) Furthermore, we have characterized D_k^s in such a manner that we could have, $v \ \varepsilon \ D_k^s$, $r \ \varepsilon \ D_k^s$ as $v \leq r$. In this case v would be redundant. In practical applications such points would be eliminated.

5) Note that P^s is always feasible as $y_k^* \ \varepsilon \ Y_k^s$ for all k and $\sum_{k=1}^{m} y_k^* + w \ \varepsilon \ X$.

6) Actually, the discrete case is easier to handle in some

ways. The center does not need to impose on the firm the requirement that $(q_k^s - y_k^s)$ belongs to C_k. Furthermore, we can guarantee convergence in a finite number of iterations.

7) A similar phenomenon would appear in cases where some firms always produce under increasing returns to scale in the relevant range; we know that only firm will produce each produce and the quantity-quantity algorithm does not take this fact into account.

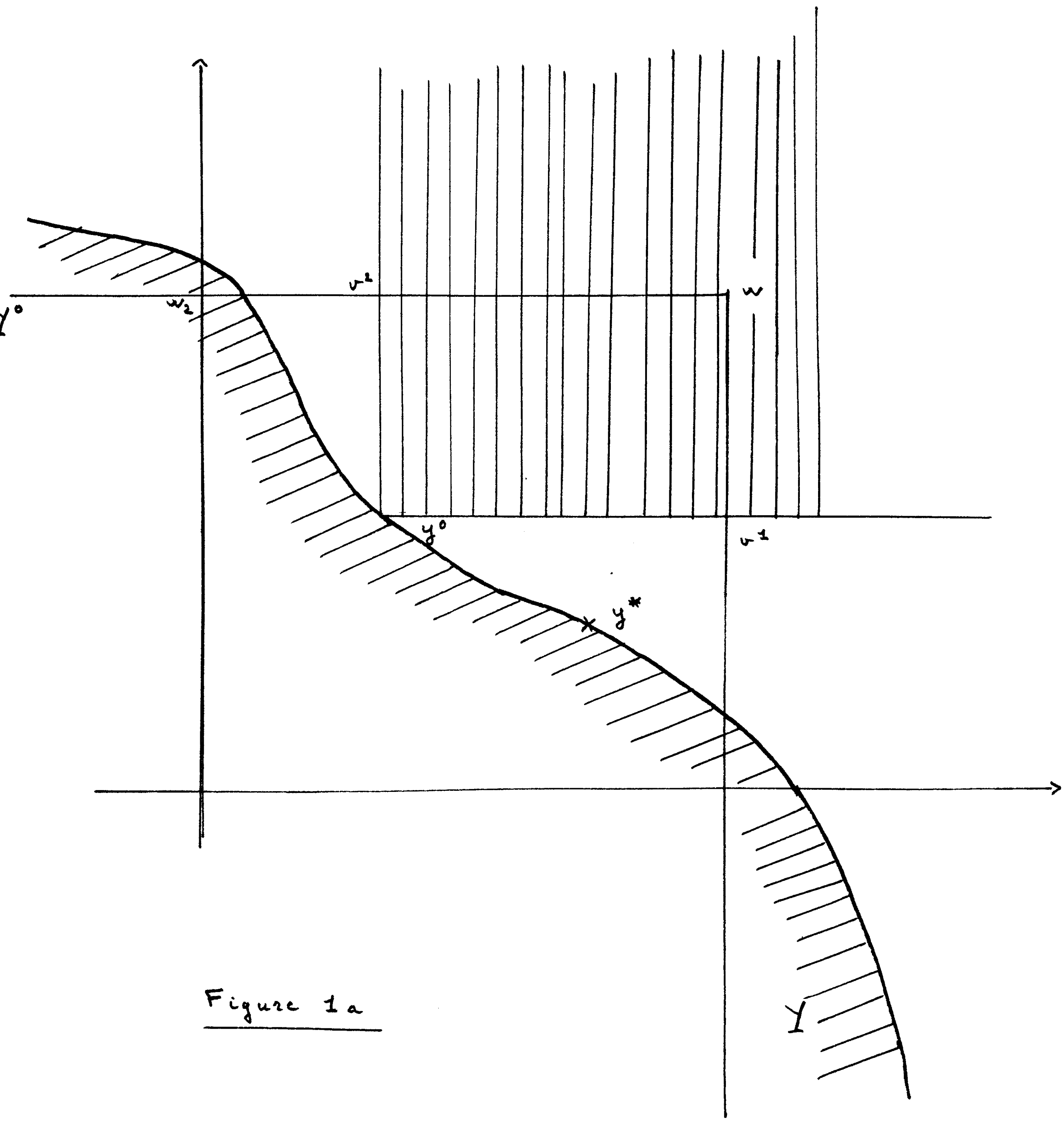


Figure 1a

Figure 1 b

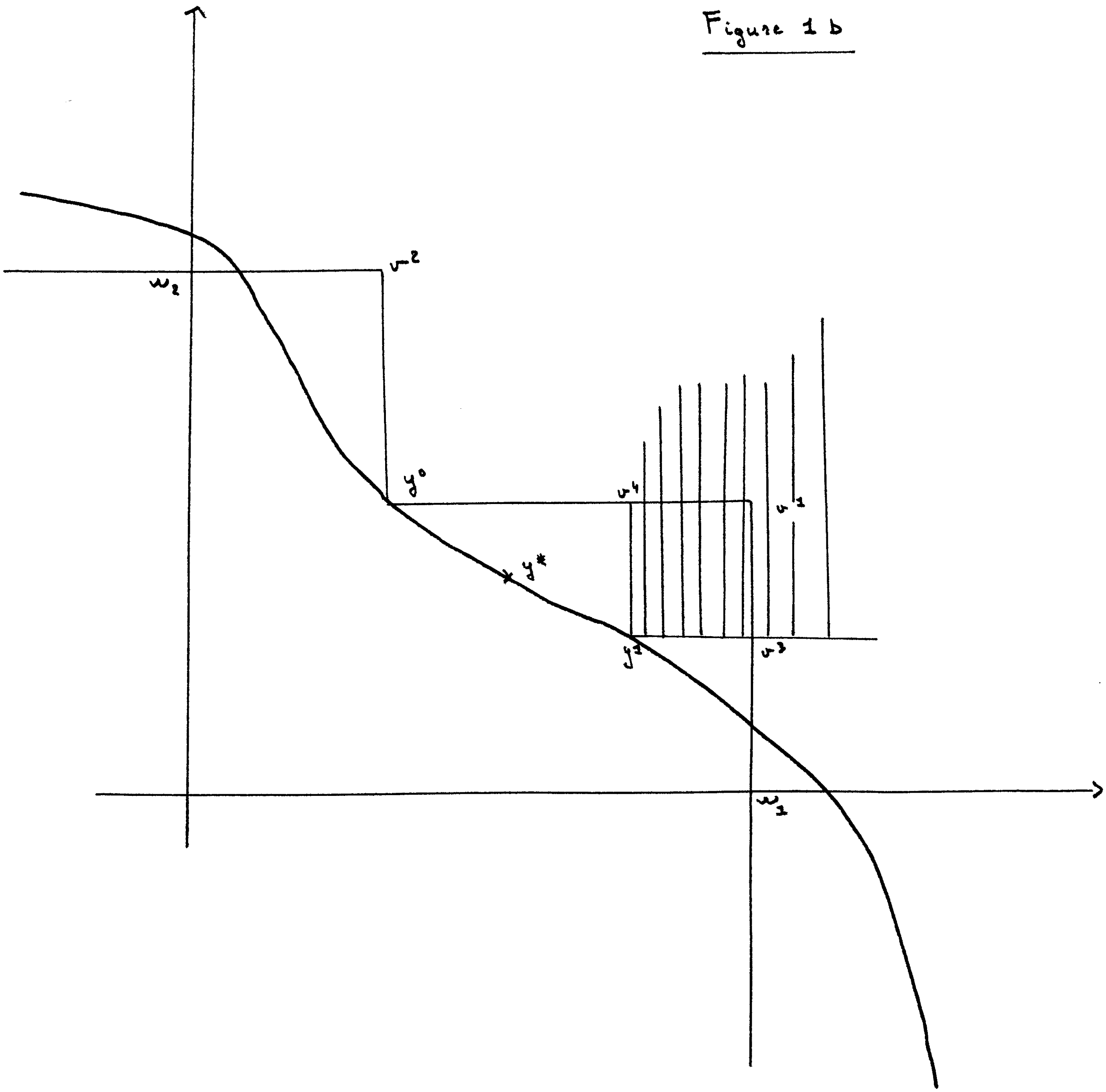
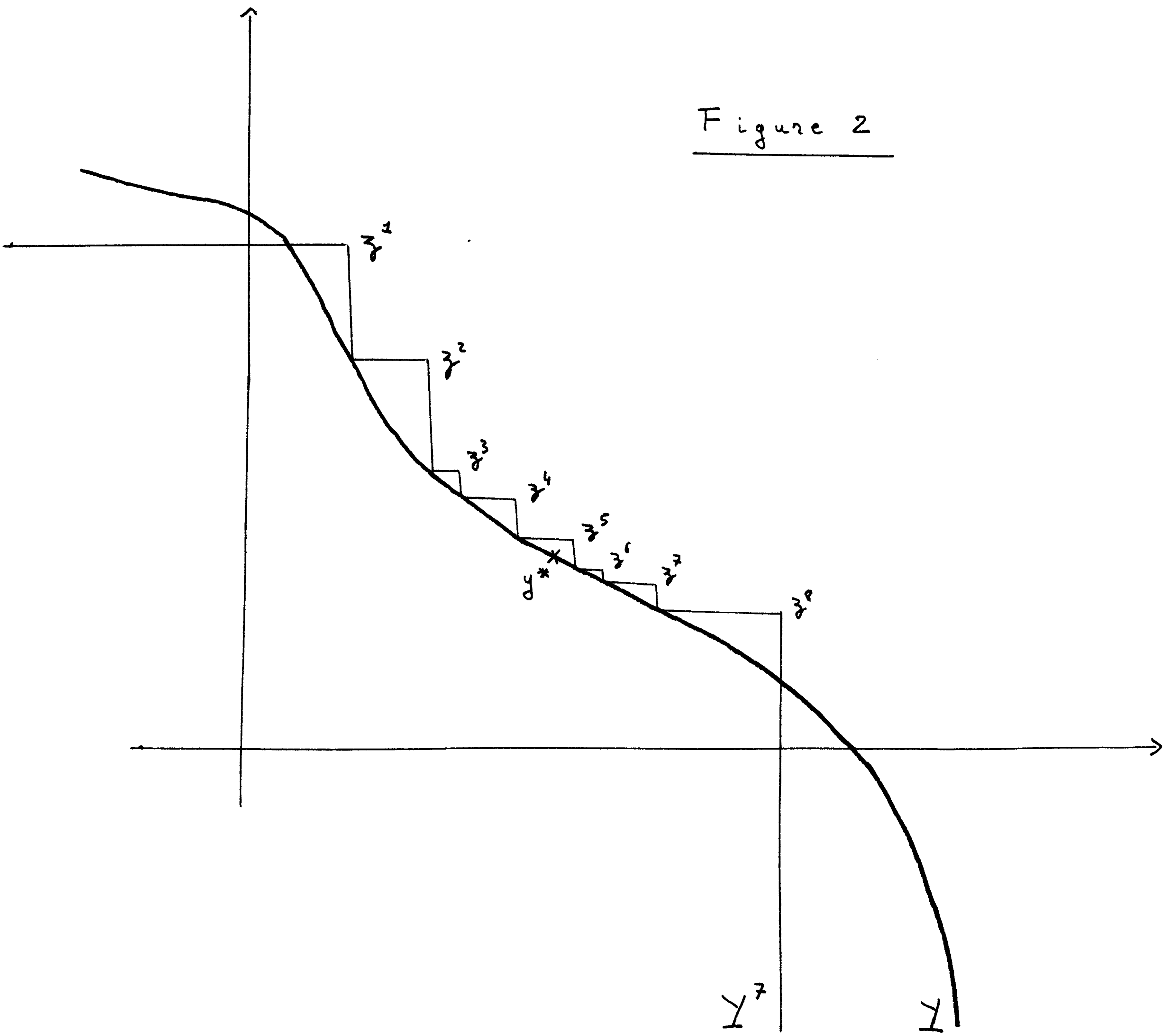


Figure 2



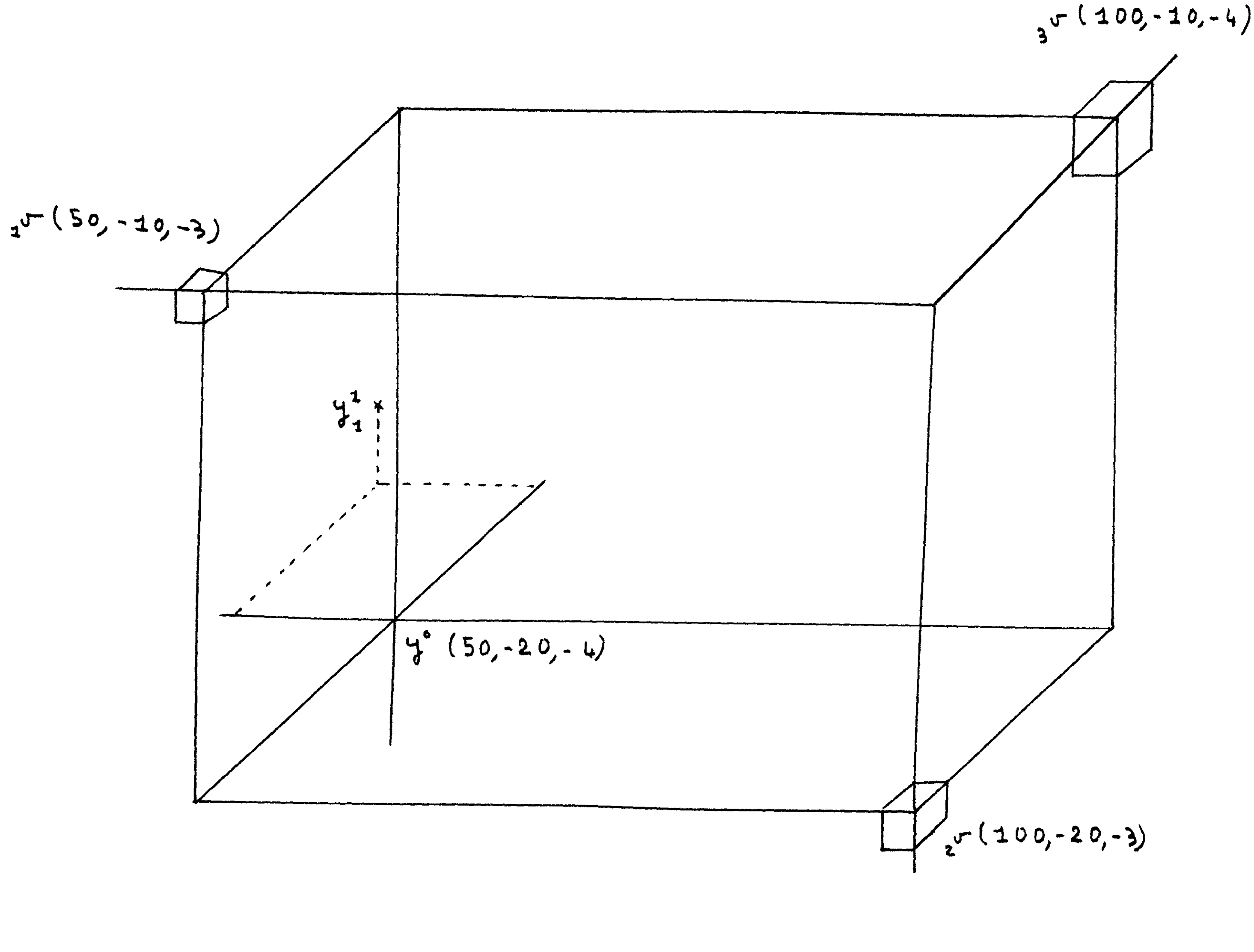


Figure 3

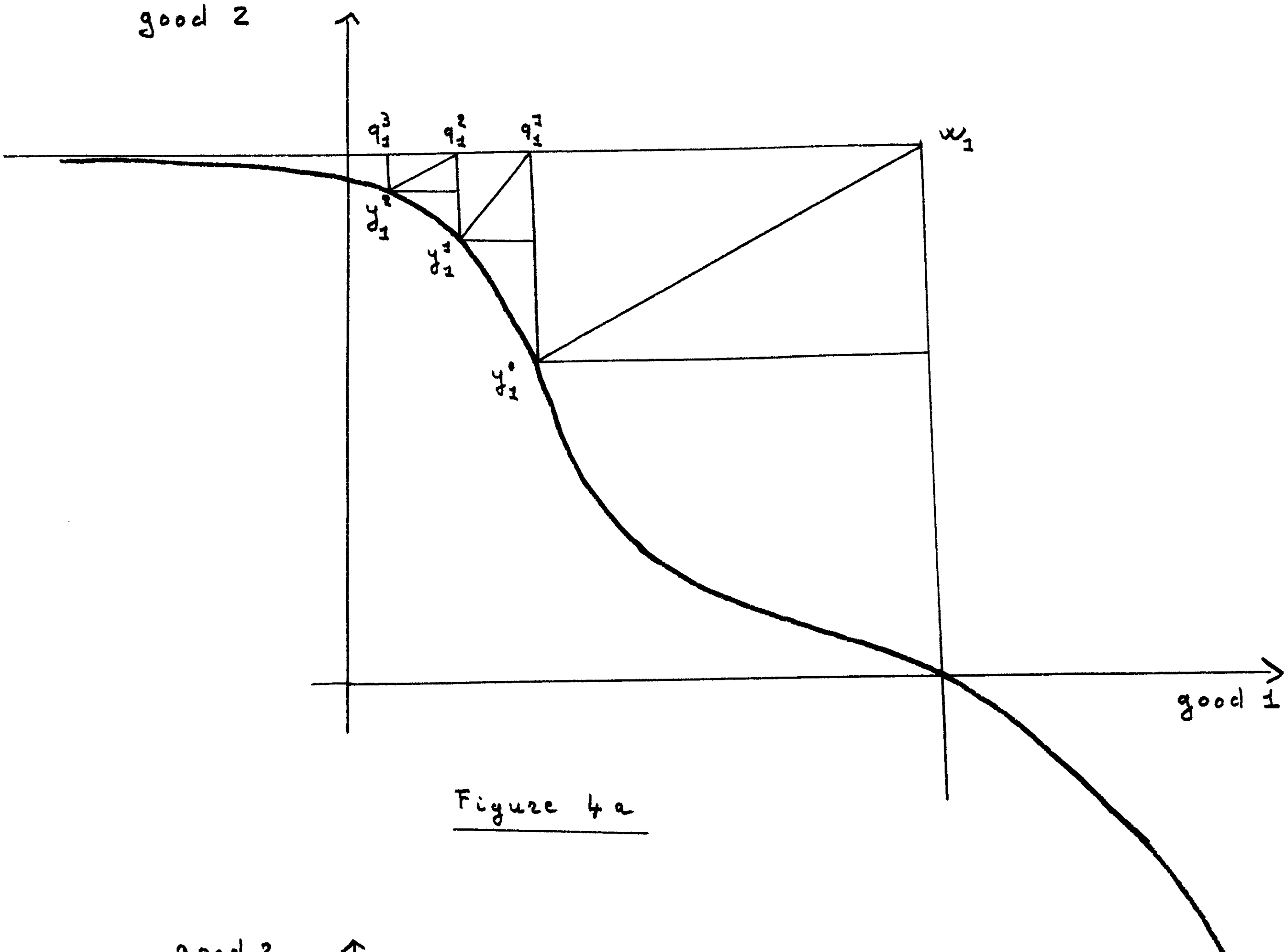


Figure 4 a

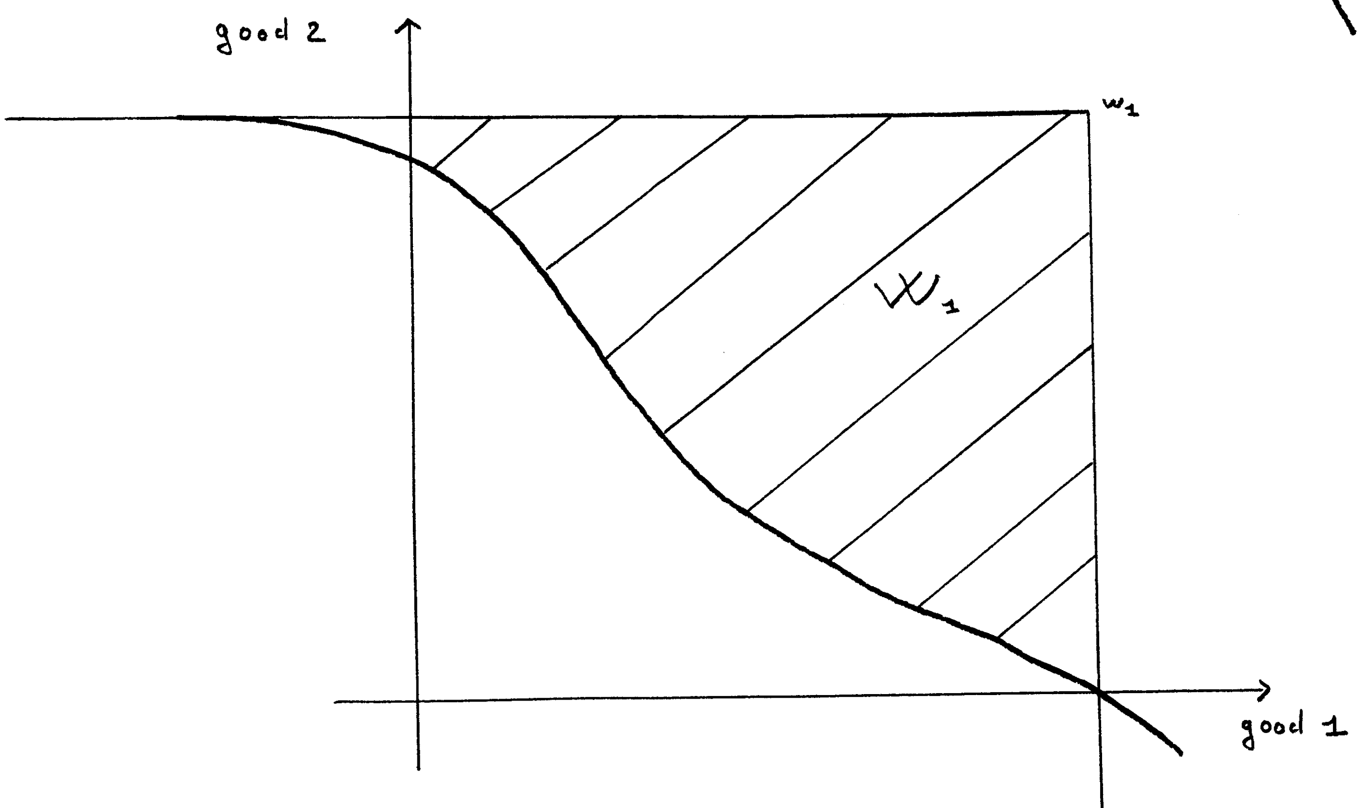


Figure 4 b

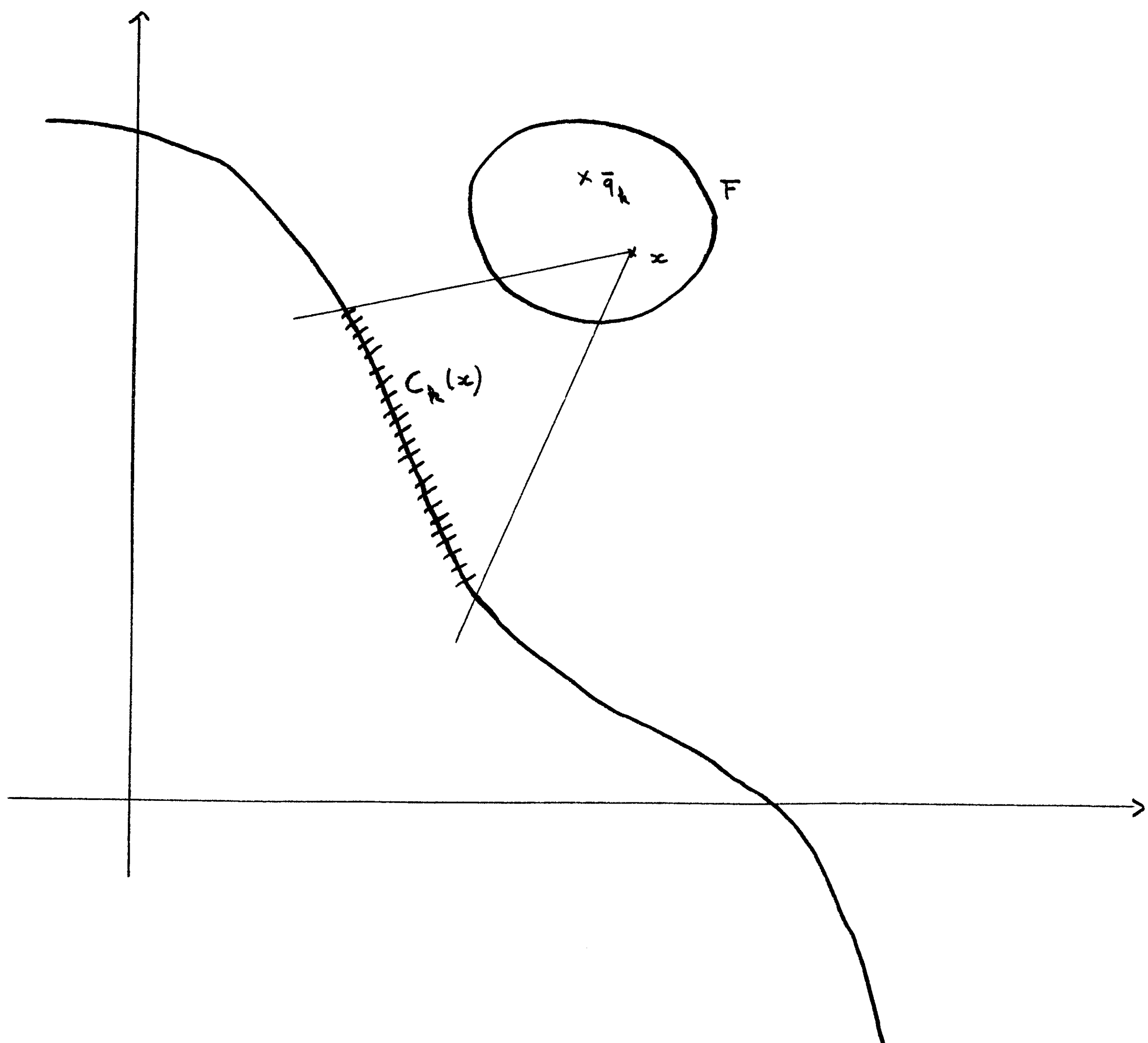


Figure 5

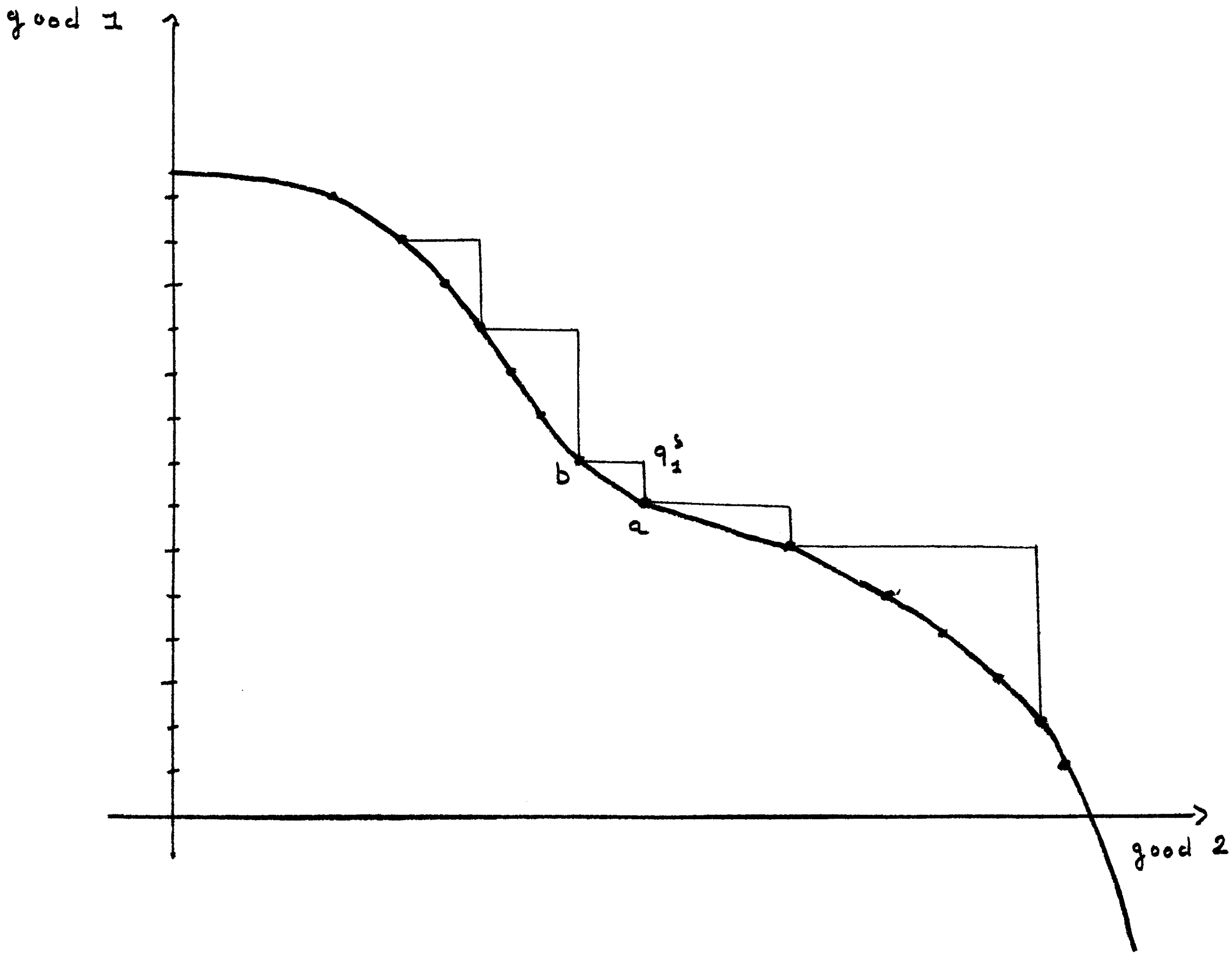


Figure 6

[1] Arrow, K.J. and L. Hurwicz (1960), "Decentralization and Computation in Resource Allocation" in R.W. Pfouts (ed.). Essays in <u>Economics and Econometrics</u>, University of North Carolina Press, Chapel Hill, 34-104.

[2] Cremer, J.C. "An Algorithm for Planning Under Increasing Returns to Scale." Unpublished Ph.D. dissertation, M.I.T.

[3] Heal, G.M. (1969). "Planning Without Prices." <u>Review of Economic Studies</u>, 36, 347-362.

[4] Lange, O. (1938). "On the Economic Theory of Socialism" in O. Lange and F.M. Taylor <u>On the Economic Theory of Socialism</u>, University of Minnesota Press, Minneapolis, 57-143.

[5] Malinvaud, E. (1967). "Decentralized Procedures for Planning" in Bacharach and Malinvaud (eds.) <u>Activity Analysis in the Theory of Growth and Planning</u>, MacMillan, London, 170-208.

CHAPTER II

I. INTRODUCTION

In this chapter we will discuss the applicability to operations research problems of the algorithm we presented in the first chapter in a planning framework.

First, we will specialize problem P of chapter I to a one firm economy. This will give an insight into the applicability of our procedure to a relatively specialized class of O.R. problems. Later, we will show that, through changes of variables, a large number of non-convex problems can be solved by our procedure.

In the third section of this chapter we will examine some applied problems for which treatment by classical methods is not appropriate and show how they can be solved by the algorithms presented in the preceeding sections. Finally, I present some very incomplete evidence on the computational efficiency of this algorithm.

II. THEORY

A. The simple form

Consider the following optimization problem, which we shall call $\overline{P}$:

$$\overline{P}: \quad \begin{aligned} &\max\ u(x) \\ &\text{s.t.}\quad f(x) \leq 0 \\ &\qquad\quad g(x) \geq 0 \end{aligned}$$

where u, f and g are functions of R^n into R, R^{n_f} and R^{n_g}, respectively; $n_f \geq 1$, $n_g \geq 0$. We assume that u, f and g are non-decreasing in all components, that u and g are upper semi-continuous and that f is lower semi-continuous.

For simplicity, we will also assume that the set of points x satisfying $f(x) \leqq 0$ and $g(x) \geqq 0$ is compact and non empty. The compactness assumption can be relaxed.

We want to show that $\overline{P}$ is but a special form of problem P of the first chapter. For this, define Y as the set of points which satisfy $f(x) \leqq 0$; formally:

$$Y = \{x | f(x) \leqq 0\} \tag{1}$$

Because f is lower semi-continuous Y is closed and therefore satisfies A1. It also clearly satisfies A2 and A3.

In a similar fashion, we can define X:

$$X = \{x | g(x) \geqq 0\} \tag{2}$$

X is closed and satisfies A6, the assumption of non-satiation.

We can therefore rewrite $\overline{P}$ under the form P':

$$P' \qquad \begin{aligned} &\max\ u(x) \\ &\text{s.t.}\quad x \in Y \\ &\qquad\quad x \in X \end{aligned}$$

P' is of the form of problem P of chapter 1, specialized to the one firm case with $\omega = 0$.

We will now quickly review the solution algorithm to show how close it is to a branch and bound algorithm.

At stage s we know a finite set D^s such that all points of X Y which are possible solutions are smaller than some $v \in D^s$.[1/] Let us define N(v) as the subset of points of X Y which are smaller or equal to v:

$$N(v) = \{x | x \leq v, \ x \in X \quad Y\} \tag{3}$$

We have divided the feasible points into a finite number of sets. For all $x \in N(v)$ $u(v)$ is an upper-bound of $u(x)$. This is the bounding operation.

We then choose the v which maximizes u(v), and divide N(v) into n subsets; this is the branching operation.

This algorithm falls within the general description of branch and bound algorithms (see Mitten [4]). However, it is original in two respects: no variable is discrete and feasible points might belong to several branches.

A method for computing a lower bound will be developed later.

In the first chapter we assumed that the vector $(q^s - y^s)$ belonged to a cone C. In this chapter C will be defined as the set of vectors of the form λe, $\lambda \geq 0$, where e is any strictly positive vector.[2/]

1/ As there is, so to speak, only one firm we disregard the firms' indices. D^s should be written D^s_1. In a similar fashion, we will use symbols y^s, q^s, Y^s.

2/ The choice of e might influence the speed of convergence.

B. <u>The general form</u>

(1) <u>An example:</u>

Consider problem L:

$$\text{L} \qquad \begin{aligned} &\max\ u(x) \\ &\text{s.t.}\quad f(x) \leqq 0 \\ &\qquad\quad\ g(x) \geqq 0 \end{aligned}$$

where $u(x) = u_1(x) - u_2(x)$ with both u_1 and u_2 continuous and non-decreasing in all components. $u(x)$ may not be monotonic so that this problem cannot be solved by the method presented earlier.

For simplicity we will assume that Y X is compact where Y and X are defined by (1) and (2), respectively.

We will introduce a new variable, y, and a function $\bar{u}$ such that $\bar{u}(x,y) = u_1(x) + y$. Note that:

$$\bar{u}(x_1, - u_2(x)) = u(x) \tag{4}$$

Consider problem L'

$$\begin{aligned} &\max\ \bar{u}(x,y) \\ &\text{s.t.}\quad f(x) \leqq 0 \\ &\qquad\quad\ u_2(x) + y \leqq 0 \\ &\qquad\quad\ g(x) \geqq 0 \end{aligned}$$

L' is a problem of the simple form we have studied earlier. We will prove that, if $(\bar{x},\bar{y})$ is a solution of L', $\bar{x}$ is a solution of L.

In order to do so let us define:

$$\bar{X} = \{(x,y) \mid f(x) \leqq 0,\ u_2(x) + y \leqq 0\} \tag{5}$$

$$\bar{Y} = \{(x,y) \mid g(x) \geqq 0\} \tag{6}$$

Obviously, we have:

$$(x,y) \in \bar{X} \qquad x \in X \tag{7}$$

$$\text{and} \quad (x,y) \in \bar{Y} \qquad y \in X \tag{8}$$

By (5), as $(\bar{x},\bar{y})$ belongs to $\bar{X}$:

$$\bar{y} \leq -\, u_2(\bar{x}) \tag{9}$$

and therefore

$$\bar{u}(\bar{x},\bar{y}) = u_1(\bar{x}) + \bar{y} \leq u_1(\bar{x}) - u_2(\bar{x}) = u(\bar{x}) \tag{10}$$

(7) and (8) imply that $\bar{x}$ is feasible for L and therefore that, if x* is a solution of L, $u(\bar{x}) \leq u(x^*)$. With (10) this implies

$$\bar{u}(\bar{x},\bar{y}) \leq u(\bar{x}) \leq u(x^*) \tag{11}$$

We will now show that all the inequalities in (11) can be replaced by equalities. It is easy to see that $(x^* - u_2(x^*))$ belongs to $\overline{X}$ $\overline{Y}$. With (4) this implies:

$$u(x^*) = \overline{u}(x^*_1 - u_2(x^*)) \leq \overline{u}(\overline{x},\overline{y}) \tag{12}$$

(11) and (12) imply:

$$u(\overline{x}) = u(x^*) \tag{13}$$

which is the equality we wanted to prove. The reader may notice that we have proved a little more than what we were set out to do: we actually proved the equivalence of L and L' (if x* is a solution of L, $(x^*_1 - u_2(x^*))$ is a solution of L').

(2) <u>The general case</u>:

Obviously we could treat similar examples where f, for instance, would satisfy $f(x) = f_1(x) - f_2(x)$, with both f_1 and f_2 non-decreasing in all components. We can, however, tackle directly a more general case.

Assume that u, for instance, is such that we can write $u(x) = \overline{u}(\lambda_u(x), \mu_u(x))$ where λ_u is a function of R^n into R^{ℓ_u} and μ_u a function of R^n into R^{m_u}. $\overline{u}$ and λ_u are non-decreasing in all components whereas μ_u is non-increasing in all components. We will use a vector of artificial variables y_u to provide a lower bound on μ_u. Before tackling the case of f and g a numerical example might be in order.

Consider $u(x,y) = (ax^2 + a'y^2 - \ xy)\ (bx^2 + cy^2 + 2)^{-1}$, defined on $x,y \geq 0$. We can write $\lambda_u(x,y) = (ax^2 + a'y^2)$ and $\mu_u(x,y) = (-xy,\ (bx^2 + cy^2 + 2)^{-1})$. Let μ_{u1} and μ_{u2} be the first and second component of μ_u, respectively. We have $u(\lambda_u(x,y), \mu_u(x,y)) = (\lambda_u(x,y) + \mu_{u1}(x,y))(\mu_{u2}(x,y))$. Note that other decompositions are possible. For instance: $\lambda_u(x,y) = (ax^2 + a'y^2, + y)$ and $\mu_u(x,y) = (-x,(bx^2 + cy^2 + 2)^{-1})$. This implies $\overline{u}(\lambda_u,\mu_u) = (\lambda_{u1} + \mu_{u1}\lambda_{u2})\mu_{u2}$. Back to the general case, let M be the following problem:

$$M \quad \begin{array}{l} \max \ u(x) \\ \text{s.t. } f(x) \leqq 0 \\ \quad\ \ g(x) \geqq 0 \end{array}$$

where $u(x) = \overline{u}(\lambda_u(x),\ \mu_u(x))$, $f(x) = \overline{f}(\lambda_f(x),\ \mu_f(x))$ and $g(x) = \overline{g}(\lambda_g(x),\ \mu_g(x))$. We assume that $\overline{u}$, $\overline{f}$, $\overline{g}$ and λ_i, $i = u$, f, g, are non-decreasing in all components and that μ_i, $i = u$, f, g, is non-increasing in all components. We define as usual $Y = \{x|f(x) \leqq 0\}$ and $X = \{x|g(x) \geqq 0\}$. X Y is assumed compact.

We will prove the following theorem.

<u>Theorem</u>. If $(\overline{x},\overline{y})$ is a solution of M':

$$\max \ \overline{u}\ (\lambda_u(x),\ y_u)$$

$$M' \qquad \text{s.t.} \quad \begin{aligned} \bar{f}(\lambda_f(x),\ y_f) &\leqq 0 \\ \bar{g}(\lambda_g(x),\ y_g) &\geqq 0 \\ -\mu_u(x) + y_u &\leqq 0 \\ -\mu_g(x) + y_g &\leqq 0 \\ -\mu_f(x) + y_f &\geqq 0 \end{aligned}$$

$\bar{x}$ is a solution of M. Furthermore, if x* is a solution of M there exists a $y^* = (y^*_u, y^*_f, y^*_g)$ such that (x^*, y^*) is a solution of M'/

<u>Proof</u>: Let us define $\bar{Y} = \{x,y \mid \bar{f}(\lambda_f(x),\ y_f) \leqq 0,\ -\mu_u(x) + y_u \leqq 0,\ -\mu_g(x) + y_g \leqq 0\}$ and $\bar{X} = \{x,y \mid \bar{g}(\lambda_g(x),\ y_g) \geqq 0,\ -\mu_f(x) + y_f \geqq 0\}$.

If (x,y) belongs to $\bar{Y}$ $\bar{X}$ we have $f(x) = f(\lambda_f(x),\ \mu_f(x)) \leqq \bar{f}(\lambda_f(x),\ y_f) \leqq 0$ and $g(x) = \bar{g}(\lambda_g(x),\ \mu_g(x)) \geqq \bar{g}(\lambda_g(x),\ y_g) \geqq 0$, and therefore x belongs to Y X

Let $(\bar{x},\bar{y})$ be a solution of M'. As $\bar{x}$ belongs to Y X, for x solution of M, we have

$$u\ (\bar{x}) \leq u(x^*) \tag{14}$$

This inequality is similar to inequality (10).
We now prove that we can transform (14) into an equality.

In order to do so note that $(x^*, \mu_u(x^*), \mu_f(x^*), \mu_g(x^*))$ belongs to $\overline{Y}\ \ \overline{X}$. We have:

$$u(x^*) = \overline{u}(\lambda_u(x^*), \mu_u(x^*)) \leq \overline{u}(\lambda_u(\overline{x}), \overline{y}_u)$$

$$\leq \overline{u}(\lambda_u(\overline{x}), \mu_u(\overline{x})) \leq u(\overline{x}) \qquad (15)$$

and therefore by (15) and (14) we have proved that:

$$u(\overline{x}) = u(x^*) \qquad (16)$$

This shows that if $(\overline{x}, \overline{y})$ is a solution of M', $\overline{x}$ is a solution of M. The converse is obvious from the proof.

Through changes of variables such as those discussed we can make most O.R. problems solvable; of course, as the number of artificial variables increase the speed of convergence is probably reduced.

III. APPLICATION

In the next few pages I would like to present some applied problems for which the quantity - quantity algorithm could be useful. In the operations research literature I have selected some problems where increasing returns to scale are present. In order to solve them the authors had to compromise.

In some cases they restricted the formulation so that techniques like geometric or integer programming could be used. Other problems were solved by gradient methods which only guarantee convergence to local optima.

A. The simple form

(1) Allocation of Inputs

In [1] DeJanvry proposes a general production function for farms:

$$Y = A \prod_{i=1}^{n} x_i^{f_i(x)} \; e^{g(x)}$$

where Y is the output and $x = (x_i)$ the vector of inputs.

Consider a farm for which all factors of production, except fertilizers, are fixed. We assume that the farmer has a limited budget to buy those fertilizers. His problem can be formalized as follows:

$$\max Y$$

$$\text{s.t.} \quad \sum_{t=1}^{n} P_i X_i \leq B$$

$$X_i \geq 0 \qquad i=1,\ldots,m$$

which can be solved by the simplest version of the algorithm. Note that the complexity of the objective function does not cause any problem.

(2) Treatment of Pollutants

The next example can also be considered as an allocation of inputs problem. However, it will show how such problems can arise in a more realistic setting. The discussion is adopted from Echer [2].

Consider a segment of river from a point A to a point B (see Figure 2.1). There are n discharges of pollutants between A and B. They will be numbered from 1 to n, with the first discharge at A. The i'th stretch will be the body of water between the i'th and (1 + i'th) discharge.

The level of pollution of water can be quantified in terms of dissolved oxygen (D.O.). The D.O. in the first stretch will depend on the purification apparatus in the first discharge. We assume they are k_1 techniques of purification applicable to the waste of this plant.

Let P_{1i} be the level at which the i'th treatment is functioning. The pollution in the first stretch can be written $f_1(P_{11},\ldots,P_{1i},\ldots,P_{1k_1})$ where f_1 is non-increasing in all components.

We will assume that the legislative authority has imposed a maximum admissible level of D.O. in the first stretch, α_1. We can write our first constraint:

$$f_1(P_{11},\ldots,P_{1i},\ldots,P_{1k_1}) \leq \alpha_1$$

Pollution in the second stretch comes from two sources. Part is carried over from the first stretch, part comes from the second discharge. With an obvious notation we can write the second constraint:

$$b_{12}f_1(P_{11},\ldots,P_{1k_1}) + f_2(P_{21},\ldots,P_{2k_2}) \leq \alpha_2$$

And through a similar reasoning the i'th constraint can be written:

$$\sum_{j=1}^{i-1} b_{ji}\, f_j(P_{j1},\ldots,P_{jk_j}) + (P_{i1},\ldots,P_{ik_i}) \leq \alpha_i$$

The cost of the purification complex at the i'th discharge can be written $c_i(P_{i1},\ldots,P_{ik_i})$. Our objective is to minimize the total cost of meeting the standards. Therefore the objective function is $\sum_{i=1}^{n} c_i(P_{i1},\ldots,P_{ik_i})$, which we are trying to minimize.

In summary we have shown that the problem to solve is the form:

$$\min. \sum_{i=1}^{n} c_i(P_{i1},\ldots,P_{ik_i})$$

$$\text{s.t.} \quad f_1(P_{11},\ldots,P_{1k_1}) \le \alpha_1$$

$$b_{12}f_1(P_{11},\ldots,P_{1k_1}) + f_2(P_{21},\ldots,P_{2k_2}) \le \alpha_2$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\sum_{j=1}^{n-1} b_{jn}f_j(P_{j1},\ldots,P_{jk_j}) + f_n(P_{n1},\ldots,P_{nl_n}) \le \alpha_n$$

$$P_{ki} \ge 0 \qquad \forall\ k,\ i$$

In his paper [2] Echer had to specialize the formulation of this problem so as to make it solvable by the methods of geometric programming.

B. General Form

(1) Engineering Design

We will take a very simple case of a general problem and show how we can use our algorithm to solve it. The discussion carries over without difficulty to more complicated systems.

To produce a certain product three operations are necessary. Those operations are not independent; the output of the first two operations can be of different quality, influencing the functioning of the process which is next in line.

Jen and al [3] give the example of an oxygen production system where the three machines are (i) an oxygen production unit, (ii) an inventory processing unit, and (iii) an inventory storage unit.

The vector of inputs of process one is denoted I_1. It includes not only the average flow of daily inputs but also the capital costs, maintenance, etc. Its output is Y_1. We have $Y_1 \leq f_1(I_1)$ where f_1 is increasing in all components (note that Y_1 might be a vector).

The second process uses both Y_1 and I_2 as inputs; its output Y_2 satisfies $Y_2 \leq f_2(I_2, Y_1)$.

Finally, the third process produces $Y_3 \leq f_3(I_3, Y_2)$. Y_3 is the final output of the system; we assume that we need a minimum amount so that we have $Y_3 \geq \overline{Y}_3$.

The problem can be written:

$$\begin{aligned} &\text{min. } c(I_1, I_2, I_3) \\ &\text{s.t. } Y_1 \leqq f_1(I_1) \\ &\qquad Y_2 \leqq f_2(I_2, Y_1) \\ &\qquad Y_3 \leqq f_3(I_3, Y_2) \\ &\qquad Y_3 \geqq \overline{Y}_3 \end{aligned}$$

which can be rewritten in the simple form:

$$\min \quad c(I_1, I_2, I_3)$$

$$\text{s.t. } 0 \leqq f(I_1) + Y'_1$$

$$0 \leqq f_2(I_2, Y_1) + Y'_2$$

$$\bar{0}_3 \leqq f_3(I_3, Y_2)$$

$$Y'_1 + Y_1 \geqq 0$$

$$Y'_2 + Y_2 \geqq 0$$

We can also use more sophisticated utility functions. For instance, assume that instead of constraining final output to be greater than some mimimum we assign a value to Y_3, $V(Y_3)$. The objective function is transformed into $V(Y_3) - c(I_1, I_2, I_3)$, which we want to maximize.

Similar problems have been treated through geometric programming or gradient methods; the latter only guaranties convergence to local minimum when the functions f are not convex.

(2) Capital Planning

Problems similar to the preceding one sometimes arise in economy-wide planning. A simple example is given by Westphal [5].

He assumes that an agency is trying to plan two interrelated industries which produce steel (good 2) and machines (good 1). The demand D_i for machines is exogeneous. The demand is met through manufacturing the machines locally, in quantity x_1, or importing them, in quantity m_1. Steel has an exogeneous demand, D_2, and is also used for building the machines. This latter part of the demand is denoted $i(x_1)$.

The supply of steel comes from two sources: domestic production (x_2) and importation (m_2).

The rest of the notation is obvious; the problem can be written:

$$\min \quad c_1(x_1) + c_2(x_2) + w_1 m_1 + w_2 m_2$$

$$\text{s.t.} \quad x_1 + m_1 \geq D_1$$

$$x_2 + m_2 \geq D_2 + i(x_1)$$

$$x_1 \geq 0$$

$$x_2 \geq 0$$

$$m_1 \geq 0$$

$$m_2 \geq 0$$

As i is increasing in all components we need one artificial variable to take care of it, and transform the problem to the simple form.

To treat this problem by known nethods Westphal has to restrain his cost functions to be piecewise linear. Our approach allows us to use a much broader family of functions. Of course the difference in costs of computation might well overcome this advantage.

IV. COMPUTATIONS

A. Computational Methods

In the two preceding sections of this chapter we have given examples of problems which our algorithm could help to solve. In the next pages we will show how the computations can be made easier - all the discussions will deal with the simple form but can be applied to the general form after the change of variables.

(1) Eliminating Irrelevant Points

First, a somewhat obvious remark. At stage s we must solve:

$$\max u(x)$$

$$\text{s.t.} \quad x \in D^s$$

$$x \in X$$

This problem is similar to the problem of Chapter I. Clearly, as the same set X is used at each stage, we need only check every time a new point is created whether it belongs to X and disregard it if it does not.

There is another more interesting method for eliminating points (the two are obviously not mutually exclusive). At every stage we have to find a point $y^s \in Y$. If y^s also belongs to X, $u(y^s)$ provides us with a lower bound on $u(x^*)$. Let $\overline{u}^s$ be the maximum for t strictly smaller than s of $u(y^t)$.

(a) If y^s is feasible (i.e., belongs to X) and $u(y^s)$ is greater than $\bar{u}^s$ we have a new lower bound on $u(x^*)$. As u is non-decreasing we know that no $v \in D^{s+1}$ such that $u(v) < \bar{u}^s$ can be greater or equal than x^*. Therefore all points of D^{s+1} which do not satisfy $u(v) > \bar{u}^s$ can be eliminated from further consideration. We make $\bar{u}^{s+1} = u(y^s)$.

(b) If y^s is not feasible or if $u(y^s)$ is smaller or equal to $\bar{u}^s$, it is sufficient to ensure that the "new points" v of D^{s+1} satisfy $u(v) \geq \bar{u}^s$. We have $\bar{u}^{s+1} = \bar{u}^s$.

This is the lower bounding mentioned above.

(2) Elimination of Redundant Points

It might happen in the course of our computations that two points v and v' of D^s satisfy $v < v'$. In this case v does not play any useful role in the algorithm and might only lead to more complicated computations.

For instance in the first chapter we examined an example where a point $y^1 = (40, -15, -6)$ is smaller than two points ${}_1v = (50, -10, -3)$ and ${}_3v = (-100, -10, -4)$ of D^1. One of the points of D^2 will be obtained by replacing the first coordinate of ${}_1v$ by the first coordinate of y^2 to give the point $(40, -10, -3)$, which we will call z_1. Similarly, when we replace the first coordinate of ${}_3v$ by the first coordinate of y^1, we obtain $z_2 = (40, -10, -4)$ which will also belong to D^2. We have $z_2 < z_1$.

Therefore we will define L^s as the subset of D^s which is maximal for the preordering $\leqq$. The construction of L^{s+1} from L^s is not much more difficult than constructing D^{s+1} from D^s. We must simply check that the "new points" are not greater than each other, or smaller than the points of L^s which are greater or equal, but not strictly greater, than y^s.

(3) Modified Objective Function

Figure 2.2 illustrates a difficulty which might arise during computations. Point ${}_1v$ is generated at the first stage. Its utility is very close to that of x*; and therefore it will be eliminated very late in the algorithm. This would not be of any consequence if the algorithm could be run until complete convergence. However, if we want to stop computations after a finite time, it is important to know that no point smaller than ${}_1v$ can be a solution.

In order to mitigate the problem mentioned above it is possible to use the following trick: Instead of using u(x) as the objective function we can use $u'(x) = u(x) + d(x)$ where d(x) is some distance of x from Y. Furthermore, we keep $\bar{u}^s = \max u(y^t)$, instead of max u'(yt), as the lower bound. On figure 2.2 this would make ${}_1v$ be chosen when solving P^s at a relatively early stage. Keeping $\bar{u}^s$ as the lower bound assures that the same difficulties will not arise with u'.

Of course, other improvements are certainly possible. In particular many of the computational improvements which have been developed for branch and bound algorithms should find an application here.

B. Computational Experiments

In Table 2.1 I present the results of four very simple computational experiments. For all of them the problem was of the form:

$$\max u(x)$$

$$\text{s.t.} \sum_{i=1}^{n} x_i \leq \lambda$$

$$x_i \geq 0 \qquad i = 1,\ldots,n$$

n was equal to 3 or 4. Due to limitations in computer time it was not possible to wait for q^s to converge to a solution. However, $\overline{u}^s$ very quickly went "close" to $u(x^*)$. It is therefore probable that the algorithm could be very powerful as an approximate method.

In case an exact solution is required the following three-step procedure is probably better than running the original algorithm until convergence:

1. Run the algorithm for a relatively short period to find an approximate solution.

2. Use the gradient method to go from the approximate solution to a local optimum.

3. Check that this local optimum is indeed a solution.

The last step can be dealt with in the following manner: Let $\tilde{x}$ be a point which we think might be an optimum. If it is, $\tilde{x}$ will also be a solution of the following problem, C:

$$
\text{C} \qquad \begin{aligned} &\max\ d(x) \\ &\text{s.t.}\quad f(x) \leq 0 \\ &\qquad\quad g(x) \geq 0 \\ &\qquad\quad u(x) \geq u(\tilde{x}) \end{aligned}
$$

where d(x) is some measure of the distance from x to Y.

As a matter of fact, if $\tilde{x}$ is a solution of the original problem, it will be the only feasible point of C. This implies that C should be much easier to solve. The heuristic reason for this statement is illustrated by Figure 2.3. We know that all the points of D^s must be smaller than our original estimate of an upper bound on the solution; they must also belong to the set X. In problem C the set X is much more restrictive, which should imply that, on the average, less new points would be introduced in D^{s+1}. Computational evidence is not available to back those statements.

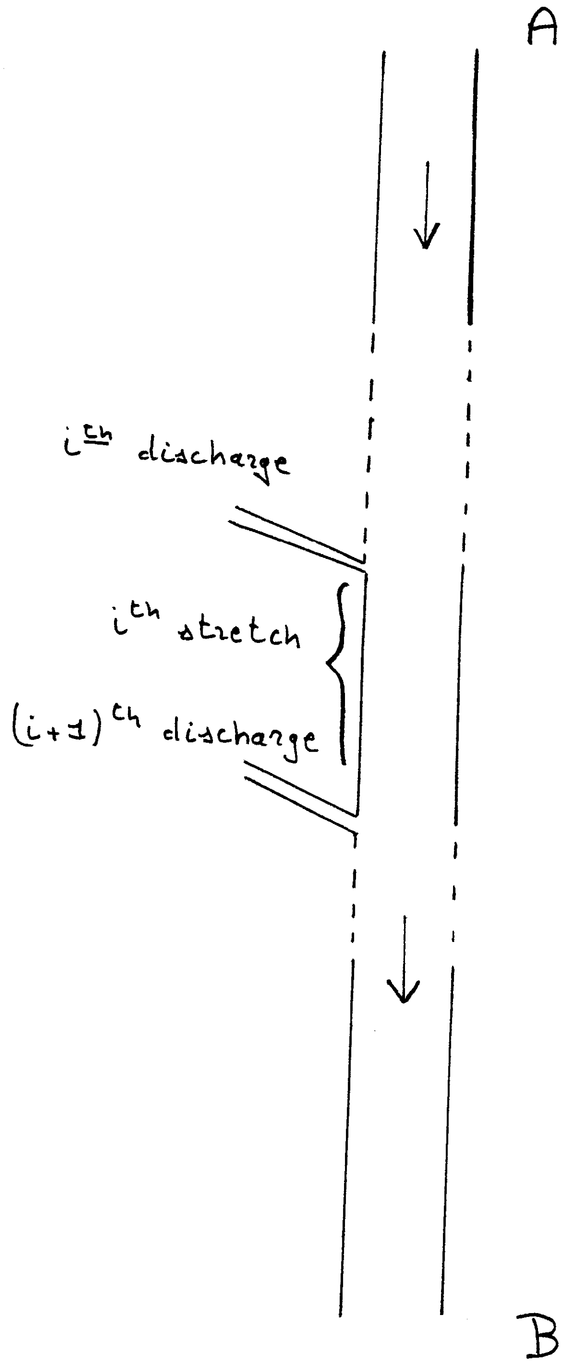


Figure 2.1

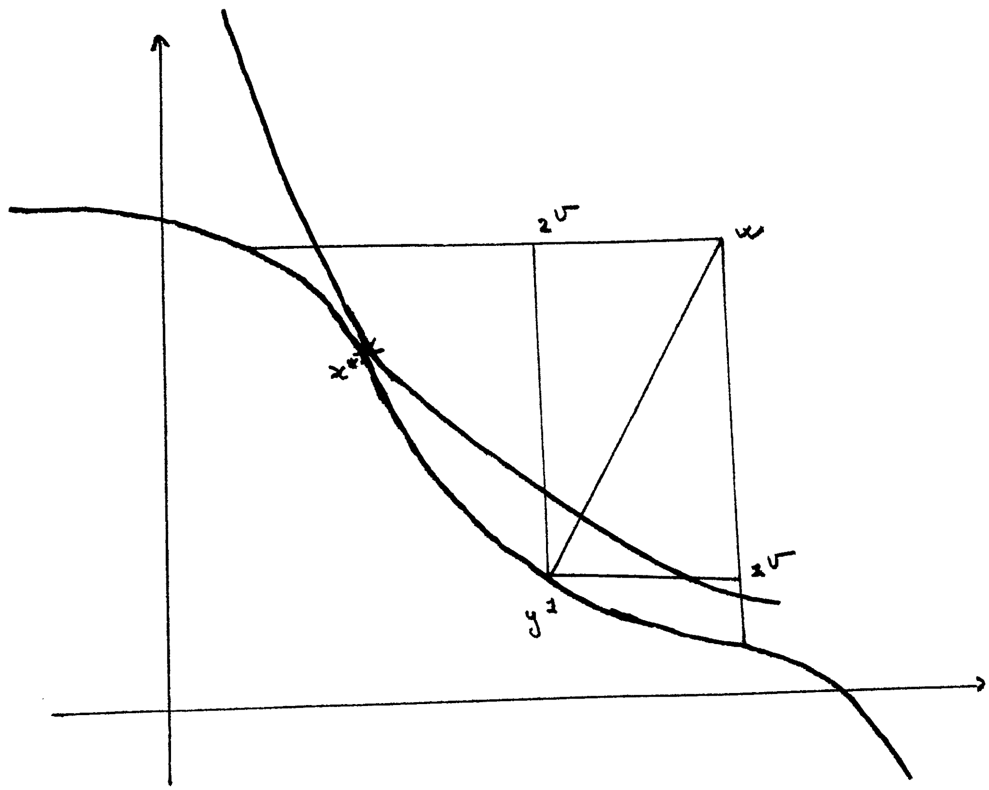


Figure 2.2

Table 2.1

	Experiment 1	Experiment 2	Experiment 3	Experiment 4
n	3	3	4	3
u	$5x_1 + 15x_1^2 + 9.5x_1x_2 + 3.5x_1x_3 + 5x_2 + x_2^2 + 6x_2x_3 + 4x_3^2$	$x_1 + 2x_2 + 3x_3 + \mathrm{pos}(x_1 - \frac{1}{2}) + \mathrm{pos}(x_1 + x_2 - \frac{1}{2}) + \mathrm{pos}(2x_1 + x_2 - 1)$	$x_2^2\sqrt{x_1} + x_3^{1.5}\sqrt{x_4} \cdot x_2x_3$	$x_1 x_2^2 x_3^3$
λ	1	1	10	1
solution to the problem	u = 8.0089	u = 3.5	u = 694.058	u = .002315
number of iterations tried	257	≃ 100	108	222
value of u at the best feasible point found	8.0077	3.4914	664.43	.002315
number of the iteration at which the best feasible point was found	239	≃ 80	100	71
number of points in D^s at that iteration	930	373	600	309

[1] De Janvry, Alain Optimal Levels of Fertilization Under Risk: Potential for Corn and Wheat Fertilization Under Alternative Price Policies in Argentina -- American Journel of Agricultural Economics, Volume 54, Number 1, February 1972, Pages 1-10.

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[3] Jen, Frank C.; Pegels, C. Carl and Dupuis, Terrence M. Optimal Capacities of Production Facilities -- Management Science, Volume 14, Number 10, June 1968, Pages 573-580.

[4] Mitten, L.G. Branch and Bound Methods: General Formulation and Properties -- Operations Research, Volume 18, Number 1, January-February 1968, Pages 24-34.

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CHAPTER III

I. INTRODUCTION

The most remarkable feature of the quantity-quantity procedure is its generality: it can be used in almost all decomposable environments which make economic sense. Therefore we can expect that in convex environments it will converge slower than specialized algorithms which take advantage of the particular properties of convex sets.[1] In most economies only a few big firms operate under increasing returns to scale. In order to integrate those firms in the planning process we need not plan all the economy with the inefficient quantity-quantity algorithm: we can use the following strategy.

The C.P.B. divides the economy in two sectors. The first of these two sectors will contain all the firms it knows to be facing decreasing returns to scale (D.R.S.). The second sector will contain all other firms in the economy. We will name it the increasing returns to scale (I.R.S.) sector. This name is not a precise description: the I.R.S. sector may contain firms producing under decreasing returns to scale without the knowledge of the planning board. The C.P.B. will use at the same time the quantity-quantity procedure in the I.R.S. sector and a more specialized algorithm in the D.R.S. sector.

This division of the economy corresponds to the practice of many governments. For instance, in many European countries, the

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[1] I have not been able to prove this conjecture. However in a few pages I will heuristically back it up in a special case.

State owns significant parts of the I.R.S. sector (i.e., railroads, electricity, etc.). It also controls rather closely the largest private firms in other industries of the I.R.S. sectors (automobile manufacturing, steel, chemistry, etc.). On the other hand communications with the firms of the D.R.S. sector are much looser.

II. DESCRIPTION OF THE ECONOMY

The economy has $m + \mu + 1$ agents: m firms in the I.R.S. sector, μ firms in the D.R.S. sector and the central planning board.

The symbols which refer to firms of the D.R.S. sector will be chosen from the Greek alphabet. The P.P.S. of firm κ (not to be confused with k), $\kappa=1,\ldots,\mu$, is closed, convex, bounded from above and satisfies free disposal.

As we discussed earlier, no assumption whatsoever is made about the returns to scale of the firms in the I.R.S. sector. They may be decreasing but either the center does not know it or does not want to use this knowledge. Symbols describing the firms of this sector will be chosen from the Roman alphabet. Y_k will be the P.P.S. of the k^{th} firm. For $k=1,\ldots,m$ Y_k is closed, bounded from above and satisfies free disposal.

The center tries to maximize a function u of the net output. u is non-decreasing in all components and convex, hence continuous. Note that in the first chapter we did not require convexity of the utility function.

In a similar fashion, not only does the consumption set X satisfy the non-satiation and closeness assumptions, it is also assumed to be convex.

A vector ω of initial resources is available. The center knows ω and X with certainty.

We can now write the problem facing the center:

$$\text{P2} \qquad \begin{aligned} & \max\ u(x) \\ & \text{s.t. } q = \sum_{k=1}^{m} qk \\ & q_k \in Y_k \qquad\qquad k=1,\ldots,m \\ & \theta = \sum_{\kappa=1}^{\mu} \theta_\kappa \\ & \theta_\kappa \in \Phi_\kappa \qquad\qquad \kappa=1,\ldots,\mu \\ & y + \theta + \omega \geqq x \\ & x \in X \end{aligned}$$

We have written P2 in a form where the outputs of the D.R.S. and I.R.S. sector play a symmetric role. This form is not convenient for describing the algorithm of this chapter.

Therefore, we write P2 under the form P2':

$$\text{P2'} \qquad \begin{aligned} & \max\ f(x) \\ & \text{s.t. } q_k = \sum_{k=1}^{m} q_k \\ & q_k \in Y_k \qquad\qquad k=1,\ldots,m \end{aligned}$$

where f(x) is defined as follows:

$$f(x) \equiv \max u(x + \zeta)$$

$$P_x \qquad \begin{aligned} \text{s.t.} \quad & \theta = \sum_{\kappa=1}^{\mu} \theta_\kappa \\ & \theta_\kappa \in \Phi_\kappa \qquad\qquad \kappa=1,\ldots,\mu \\ & \theta + \omega \geq \zeta \\ & x + \zeta \in X \end{aligned}$$

if P_x is feasible and $f(x) \equiv -\infty$ is P_x is not feasible.

$f(x)$ is the highest utility the C.P.B. can obtain when the net output of the I.R.S. sector is fixed at x.

It is easy to see that f satisfies assumptions A4 and A5 of the first chapter: it is non-decreasing in all components and upper semi-continuous. Hence we could solve problem P2' with the quantity-quantity algorithm if we knew the function f. In the following pages we will study two algorithms where the center conducts a version of the quantity-quantity algorithm, using in its optimizations better and better approximations to the function f. The difference between the two algorithms lies mainly in the form the information transmitted between the center and the firms.

III. A PROCEDURE USING PRODUCTION TARGETS IN BOTH SECTORS

A. Weitzman's Method For Approximating Convex Production Possibility Sets

Weitzman [2] has proposed an algorithm very similar to the quantity-quantity procedure although its use is limited to convex economies. In this section we will describe his method for approximating the P.P.S. of convex firms. This method will be used by the planning board in both our algorithms. The following discussion can be visualized in Figure 3.1.

Let Φ_κ be a convex P.P.S. and assume that at stage s the planners know a convex set Φ_κ^s such that:

$$\Phi_\kappa^s \supseteq \Phi_\kappa \tag{1}$$

A point ξ_κ^s on the upper frontier of Φ_κ^s is chosen. This point plays a role similar to that of q_k^s, the production target, in the quantity-quantity algorithm; its exact choice will be precised later.

The center asks firm κ for a κ-efficient point, smaller than ζ_κ^s. The firm transmits a point γ_κ^s, and also its marginal rate of substitution, π_κ^s, at that point.[2] By well-known properties of convex sets we have:

[2]If the production possibility surface is not smooth at γ_κ^s there might exist several hyperplanes tangent to Φ_κ at γ_κ^s. In this case any of them will do.

$$\eta \; \varepsilon \; \Phi_{\kappa} \Rightarrow \pi_{\kappa}^{s} \eta \leq \pi_{\kappa} \gamma_{\kappa}^{s} \tag{2}$$

By analogy with the construction of Y_k^{s+1} from Y_k^s in the first chapter we build Φ_{κ}^{s+1} in the following manner:

$$\eta \; \varepsilon \; \Phi_{\kappa}^{s+1} \Leftrightarrow \eta \; \varepsilon \; \Phi_{\kappa}^{s}, \pi_{\kappa}^{s} \eta \leq \pi_{\kappa}^{s} \gamma_{\kappa}^{s} \tag{3}$$

By (2) and (3) it is clear that Φ_{κ}^{s+1} is a subset of Φ_{κ}^{s}. Furthermore, if Φ_{κ}^{s} is convex so is Φ_{κ}^{s+1}.

We assume that at the origin the center had chosen a convex Φ_{κ}^{o} such that $\Phi_{\kappa}^{o} \supseteq \Phi_{\kappa}$. We have:

$$\Phi_{\kappa} \subseteq \Phi_{\kappa}^{s} \subseteq \cdots \subseteq \Phi_{\kappa}^{1} \subseteq \Phi_{\kappa}^{o} \tag{4}$$

for all s. Furthermore, the Φ_{κ}^{s} are all convex.

Why should we assume that the Weitzman-type approximations are more efficient than the quantity-quantity-type approximations? In equation (3) we have eliminated from further consideration all points belonging to the intersection of Φ_{κ}^{s} and a half plane; in the first chapter a similar operation only eliminated the intersection of Y_k^s and a quadrant. In a n-dimensional space there are 2^{n-1} quadrants in a half space. Whether this fact can be used to prove in a rigorous manner that Weitzman's procedure will converge faster than the quantity-quantity algorithm in economies where they are both applicable, I do not know. But it certainly raises a strong presumption in favor of this hypothesis.

B. The Algorithm

The C.P.B. starts with convex a priori approximations to the P.P.S. of the firms of the D.R.S. sector, Φ_κ^o is the approximation to Φ_κ, the P.P.S. of the κ^{th} firm; it satisfies $\Phi_\kappa \subseteq \Phi_\kappa^o$.

At the same time the center builds m sets Y_k^o, similar to those used in the first chapter.

At stage s = 1,..., ∞, the C.P.B. has built from information acquired in previous stages convex sets Φ_κ^s, $\kappa=1, \ldots, \mu$, such that $\Phi_\kappa \subseteq \Phi_\kappa^s \subseteq \Phi_\kappa^{s+1} \subseteq \ldots \subseteq \Phi_\kappa^o$. In a similar fashion it knows m Y_k^s such that $Y_k^s \subseteq \ldots \subseteq Y_k^o$. To each Y_k^s correspond a D_k^s such that:

$$x \in Y_k^s \Longleftrightarrow \exists v \in D_k^s \text{ such that } x \leqq v \qquad (5)$$

The center acts temporarily as if the Φ_κ^s, $\kappa=1,\ldots, \mu$, and Y_k^s, $k=1,\ldots, m$, were the real P.P.S. of the firms. It solves:

$$Q^s \quad \begin{aligned} &\max u(x) \\ &\text{s.t. } q = \sum_{k=1}^{m} q_k \\ &q_k \in Y_k^s \qquad\qquad k=1,\ldots, m \\ &\theta = \sum_{\kappa=1}^{\mu} \theta_\kappa \\ &\theta_\kappa \in \Phi_\kappa^s \qquad\qquad \kappa=1,\ldots, \mu \end{aligned}$$

$$y + \theta + \omega \geq x$$

$$x \in X$$

At first sight this problem looks unsolvable. However, we can use the reasoning of the first chapter to show that, because u is non-decreasing, at least one solution of Q^s will be such that q_k^s belongs to D_k^s for all k. q_k^s is the production target of firm k of the I.R.S. sector. Instead of solving Q^s the center can therefore solve Q'^s:

$$\max f^s(x)$$

$$Q'^s \qquad \text{s.t. } q = \sum_{k=1}^{m} q_k$$

$$q_k \in D_k^s \qquad\qquad k=1,\ldots, m$$

where: $$f^s(x) \equiv \max u(x + \zeta)$$

$$\text{s.t. } \zeta = \sum_{\kappa=1}^{\mu} \zeta_\kappa$$

$$Q_x^s \qquad \zeta_\kappa \in \Phi_\kappa^s \qquad\qquad \kappa=1,\ldots, \mu$$

$$\zeta + x \in X$$

if Q_x^s is feasible and $f^s(x) \equiv -\infty$ if Q_x^s is not feasible. f^s is the approximation to f we have mentioned above.

The problem Q^s_x is a convex problem and therefore can be solved by known methods for all $x = \sum_{k=1}^{m} v_k$, $v_k \in D^s_k$. Problem Q'^s is then a maximization problem whose domain is a finite set so that it can be solved by comparing the value taken by P^s at all the points in its domain of definition. It is clear that the computations are still formidable but at least they are feasible.[3]

Let $(q^s_1,\ldots, q^s_m)$ be a solution of Q'^s; define $q^s = \sum_{k=1}^{m} q^s_k$ and let $(\zeta^s_1,\ldots, \zeta^s_\mu)$ be a solution of $Q^s_q s$. Because u is non-decreasing we can assume that for all $\kappa=1,\ldots, \mu$, ζ^s_κ belongs to the upper frontier of Φ^s_κ. Obviously $(q^s_1,\ldots, q^s_m, \zeta^s_1,\ldots, \zeta^s_\mu)$ is a solution of Q^s.

The center proposes those q^s_k and ζ^s_κ to the firms as production targets. All firms must transmit to the center a point which is k (resp $\underline{\kappa}$) -efficient and smaller than q^s_k (resp ζ^s_κ). Let y^s_k and γ^s_κ be the answers of firms k of the I.R.S. sector and κ of the D.R.S. sector, respectively.

The firms of the D.R.S. sector are asked for more information: They must also give the vector π^s_κ of their marginal rate of substitution[4] at point γ^s_κ. There is no loss of generality in assuming that the vectors π^s_κ are normalized so that $\sum_{i=1}^{n} \pi^s_{i\kappa} = 1$ for all s and all κ.

---

[3]Actually Q^s_x does not have to be solved at each stage for all $x = \sum_{k=1}^{m} v_k$, $v_k \in D^s_k$. Most of the points of D^s_k also belong to D^{s+1}_k. So that through $f^{s-1}(x)$ we know an upper bound of $f^s(x)$ for quite a few x. This remark can be used to develop an algorithm which limits the number of problems Q^s_x to be solved at each stage.

[4]The remark of footnote 2 in case of non-uniqueness of π^s_κ applies here.

The sets Y_k^{s+1}, $k=1,\ldots, m$, and Φ_κ^{s+1}, $\kappa=1,\ldots, \mu$, are then constructed through the techniques used in the quantity-quantity and Weitzman's algorithms, respectively.

Therefore:

$$x \in Y_k^{s+1} \Longleftrightarrow x \in Y_k^s,\ x \not> y_k^s \qquad k=1,\ldots, m \tag{6}$$

$$\eta \in \Phi_\kappa^{s+1} \Longleftrightarrow \eta \in \Phi_\kappa^s,\ \pi_\kappa^s \eta \leq \pi_\kappa^s \gamma_\kappa^s \qquad \kappa=1,\ldots, \mu \tag{7}$$

We will now show that this procedure does indeed converge towards the societal optimum, as seen by the C.P.B.

C. Convergence

We take the same precautions as in the first chapter to insure that the sequence $\{q_k^s\}$ converge to a point of Y_k: For all s and all k, $(q_k^s - y_k^s)$ is assumed to belong to a closed set C_k, interior to the positive quadrant. Furthermore the set W_k of points of Y_k^o which do not belong to the interior of Y_k is assumed compact.

By this last assumption we insure that the sequences $\{q_k^s\}$ have at least a limit point. In a similar manner, the sequences $\{\zeta_k^s\}$ of production targets offered to the firms of the D.R.S. sector and the sequences $\{\gamma_k^s\}$ of their answers have at least a limit point.

Furthermore the π_κ^s have all been chosen such that $\sum_{i=1}^{n} \pi_{i\kappa}^s = 1$. Therefore they all belong to a compact set and have a limit point.

Hence we know that the $2(\mu + m)$ sequences $\{q_k^s\}$, $\{y_k^s\}$, $\{\zeta_\kappa^s\}$, $\{\pi_\kappa^s\}$ converge. By a classical result of analysis we know that there exists a subsequence S of the indices such that those $2(m + \mu)$ sequences converge simultaneously along S.

Weitzman's proof [2] of the convergence of the $\{\zeta_\kappa^s\}$ to points of Φ_κ carries here. The reader who may wish to refer to that proof should simply note that it is independent of the choices of the ζ_κ^s at each stage. They could even be chosen randomly as long as they belong to the upper frontier of Φ_κ^s for all κ and all s. Therefore the limit $\overline{\zeta}_\kappa$ of $\{\zeta_\kappa^s\}$, $s \in S$, belongs to Φ_κ for all .

$\overline{q}_k$, the limit of $\{q_k^s\}$ along S, belong to Y_k, for all k. The proof is similar to the first chapter's.

As $q^s + \zeta^s + \omega$ belongs to X for all s $(\overline{q}_1 \ldots, \overline{q}_m, \overline{\zeta}_1, \ldots, \overline{\zeta}_\mu)$ will be feasible.

Let $\overline{q} = \sum_{k=1}^{m} \overline{q}_k$. Let $(\overline{\lambda}_1, \ldots, \overline{\lambda}_\mu)$ be a solution of $P_{\overline{q}}$ and define $\overline{\lambda} = \sum_{\kappa=1}^{\mu} \overline{\lambda}_\kappa + \omega$. We have

$$f(\overline{q}) = u(\overline{q} + \overline{\lambda}) \geq u(\overline{q} + \overline{\zeta}) \tag{8}$$

where

$$\overline{\zeta} = \sum_{\kappa=1}^{\mu} \overline{\zeta}_\kappa + \omega$$

$\overline{q} + \overline{\lambda}$ is feasible so that for all s:

$$u(q^s + \zeta^s) \geq u(\overline{q} + \overline{\lambda}) = f(\overline{q}) \tag{9}$$

as $(q_1^s, q_2^s, \ldots, q_m^s, \zeta_1^s, \ldots, \zeta^s)$ is a solution of Q^s.

As u is continuous by letting s go to infinity we deduce from (9):

$$u(\overline{q} + \overline{\zeta}) \geq u(\overline{q} + \overline{\lambda}) = f(\overline{q}) \tag{10}$$

and by (8) and (10)

$$u(\overline{q} + \overline{\lambda}) = u(\overline{q} + \overline{\zeta})$$

so that $(\overline{\zeta}_1, \ldots, \overline{\zeta}_\mu)$ is a solution of $P_{\overline{q}}$.

Now we can easily show that $(\overline{q}_1, \ldots, \overline{q}_m)$ is a solution of P2'. Let z be a feasible point of $\sum_{k=1}^{m} Y_k$; for all s we have:

$$f^s(z) \leq f^s(q^s) \tag{11}$$

As $f^s(z)$ is always greater than f(z) we can also write:

$$f(z) \leq f^s(q^s) = u(q^s + \zeta^s) \tag{12}$$

For all s as $\lim u(q^s + \zeta^s) = u(\overline{q} + \overline{\zeta})$ by (12) we have:

$$f(z) \leq u(\overline{q} + \overline{\zeta}) = f(\overline{q})$$

Therefore $\overline{q}$ is a solution of P2'and $(\overline{q}_1, \overline{q}_2 \ldots q_m, \overline{\zeta}_1, \ldots, \overline{\zeta}_\kappa, \ldots, \overline{\zeta}_\mu)$ a solution of P2.

IV. A PROCEDURE USING PRICES IN THE D.R.S. SECTOR

If we want to introduce however little realism in our planning games, we must be concerned with the size of the computations that the center faces.

The easiest way to reduce their scale is probably to group similar firms together for the purpose of computations by the center. Production targets algorithms do not allow such an escape. Consider three efficient points y_A, y_B, and y_C in the P.P.S. Y_A, Y_B, Y_C of three firms A, B and C. There is no reason for $y_A + y_B + y_C$ to be efficient in $Y_A + Y_B + Y_C$. So that if, for planning purposes, we decided to aggregate A, B and C we would need an intermediate level of authority with a good knowledge of Y_A, Y_B and Y_C.

In convex economics, there exists one way out of the preceding dilemma. If y_A, y_B and y_C are chosen so that they maximize py over Y_i, i = A, B, C, for some $p > 0$ $(y_A + y_B + y_C)$ will be efficient in $Y_A + Y_B + Y_C$. This idea forms the basis of an algorithm by Malinvaud [1].

It is therefore natural to try and find an algorithm which will be price-led in the D.R.S. sector and use production targets in the I.R.S. sector.

A. Malinvaud's Approximation to a Convex P.P.S.

Let Y be a convex set. We know that for all finite family $(x_1, \ldots, x_\rho)$ of points of $Y = \sum_{j=1}^{\rho} r_j x_j$ belongs to Y for all $(r_1, \ldots, r_\rho)$ such that $\sum_{j=1}^{\rho} r_{j=1}$.

The following algorithm will use this description of a convex set to generate approximations to the P.P.S. of the firms of the D.R.S. sector. The discussion can be visualized on Figure 3.2.

Let Φ_κ be the convex P.P.S. of firm κ of the D.R.S. Assume that at stage s the center knows a convex set Ω_κ^s such that:

$$\Omega_\kappa^s \subseteq \Phi_\kappa$$

Ω_κ^o is given from a priori knowledge of the center. It might consist of only one feasible point.

Firm κ announces a point γ_κ^s which belongs to Φ_κ. For all $\zeta \in \Omega_\kappa^s$ and all $r \in [0,1]$ $\eta = r\gamma_\kappa^s + (1 - r)\zeta$ belongs to Φ_κ. It is therefore natural for the center to build Ω_κ^{s+1} as the set of all convex combinations of γ_κ^s and one point of Ω_κ^s.

$$\Omega_\kappa^{s+1} = \{\eta \mid \eta = r\gamma_\kappa^s + (1 - r)\zeta,\ r \in [0,1],\ \zeta \in \Omega_\kappa^s\}$$

Note that γ_κ^s and all the points of Ω_κ^s belong to Ω_κ^{s+1}.

B. The Algorithm

Our new algorithm is very similar to the preceding one: however at every stage the C.P.B. will use two types of approximations to the P.P.S. of the firms of the D.R.S. sector.

The Weitzman-type approximations, Φ^s_κ, $\kappa=1, \ldots, \mu$, includes Φ_κ, for all s. They will be used for computing the production targets of the firms of the I.R.S. sector. The "Malinvaud-type" approximations, Ω^s_κ, $\kappa=1, \ldots, \mu$, will be used to determine the messages to be sent to the firms of the D.R.S. sector. They will be a subset of Φ_κ for all s.

The C.P.B. starts with 2 convex approximations to the P.P.S. of the firms of the D.R.S. sector. The first one Φ^o_κ, $\kappa=1, \ldots, \mu$, is convex and satisfies:

$$\Phi_\kappa \subseteq \Phi^o_\kappa \qquad \kappa=1, \ldots, \mu \qquad (13)$$

The second approximation Ω^o_κ is also convex but instead of (13) it satisfies:

$$\Phi_\kappa \supseteq \Omega^o_\kappa \qquad \kappa=1, \ldots, \mu \qquad (14)$$

The planning board also knows an approximation to the P.P.S. of the firms of the I.R.S. sector. This approximation, Y^o_k, $k=1, \ldots, m$, is similar to the one we used in the pure quantity-quantity algorithm, in the first chapter.

At stage s the center has built for firm κ, $\kappa=1, . . , \mu$, of the D.R.S. sector two approximations of Φ_κ, Φ_κ^s and Ω_κ^s respectively. Both are convex and we have:

$$\Omega_\kappa^o \subseteq \Omega_\kappa^1 \subseteq \cdots \subseteq \Omega_\kappa^s \subseteq \Phi_\kappa \subseteq \Phi_\kappa^s \subseteq \cdots \subseteq \Phi_\kappa^1 \subseteq \Phi_\kappa^o \qquad (15)$$

We will examine later how Φ_κ^{s+1} and Ω_κ^{s+1} are built from Φ_κ^s and Ω_κ^s.

The center also knows m sets Y_k^s, $k=1, . . ., m$, similar to the Y_k^s of the quantity-quantity algorithm. For each s and k there exists a finite family D_k^s such that:

$$x \in Y_k^s \Longleftrightarrow \exists\, v \in D_k^s \text{ s.t.} \qquad x \leqq v$$

The center first uses the Φ_κ^s to compute the production targets of the firms of the I.R.S. sector. It solves problem R^s which is in every way similar to problem Q^s in the first algorithm of this chapter.

$$R^s \qquad \begin{aligned} &\max u(x) \\ \text{s.t.} \quad & q = \sum_{k=1}^{m} q_k \\ & q_k \in Y_k^s \qquad k=1, . . ., m \\ & \theta = \sum_{\kappa=1}^{\mu} \theta_\kappa \end{aligned}$$

$$\phi_\kappa \in \Phi_\kappa^s \qquad \kappa = 1, \ldots,$$

$$y + \theta + \omega \geq x$$

$$x \in X$$

We notice again that, as u is non-decreasing in all components, there exists at least one solution of R^s such that q_k^s belongs to D_k^s for all k. Therefore R^s can be written under the form R'^s, similar to Q'^s.

$$R'^s \quad \begin{cases} \max f^s(x) \\ \text{s.t. } q = \sum_{k=1}^{m} q_k \\ q_k \in D_k^s \qquad k=1, \ldots, m \end{cases}$$

where $f^s(x) \equiv \max u(x + \zeta)$

$$R_x^s \quad \begin{cases} \text{s.t. } \zeta = \sum_{\kappa=1}^{\mu} \zeta_\kappa + \omega \\ \zeta_\kappa \in \Phi_\kappa^s \qquad \kappa = 1, \ldots, \mu \\ \zeta + x \in X \end{cases}$$

if R_x^s is feasible and $f^s(x) \equiv -\infty$ if R_x^s is not feasible. $(\zeta_1^s, \ldots\ldots, \zeta_\mu^s)$ is the solution of $R^s_{q^s}$ where $q^s = \sum_{k=1}^{m} q_k^s$, and $(q_1^s, q_2^s, \ldots, q_m^s)$ is the production plan for the I.R.S. sector, solution of problem R'^s. The center will use those quantities as production targets for the I.R.S. sector. It now needs to determine the messages to be sent to the firms of the D.R.S. set. In order to so so it uses the Ω_κ^s: it solves problem T^s.

$$T^s \quad \begin{cases} \max u(q^s + \psi) & \\ \theta = \sum_{\kappa=1}^{\mu} \theta_\kappa & \\ \theta_\kappa \in \Omega_\kappa^s & \kappa=1, \ldots, \mu \\ \theta + \omega \geq \psi & \\ \psi + q^s \in X & \end{cases}$$

where $q^s = \sum_{k=1}^{m} q_k^s$.

This is a convex problem. Let $(\psi_1^s, \ldots, \psi_m^s)$ be one of its solutions. By convexity of the Ω_κ^s there exists a vector $\pi^s \geq 0$ such that, for all κ:

$$\pi^s \psi_\kappa^s \geq \pi^s \eta \text{ for all } \eta \in \Omega_\kappa^s \tag{16}$$

We will assume, without loss of generality that $\sum_{i=1}^{n} \pi_i^s = 1$ for all s. The center now uses the q_k^s, k=1, . . ., m, and π^s to obtain new information from the firms.

Firm k of the I.R.S. sector is asked for a k-efficient point smaller than q_k^s. Let y_k^s be its answer. Y_k^{s+1} is built from Y_k^s in the same manner as in the quantity-quantity algorithm:

$$x \in Y_k^{s+1} \Longleftrightarrow x \in Y_k^s,\ x \not> y_k^s \tag{17}$$

Firm κ of the D.R.S. sector is asked for a point γ_κ^s which maximizes its profits at the prices π^s, i.e.:

$$\pi^s\ \gamma_\kappa^s \geq \pi^s \cdot \eta \quad \forall \eta \in \Phi_\kappa \tag{18}$$

From γ_κ^s and π^s the center constructs Ω_κ^{s+1} and Φ_κ^{s+1}. The process can be visualized with the help of Figure 3.3. Φ_κ^{s+1} is built from Φ_κ^s in the same manner we built it in the preceding algorithm.

$$\Phi_\kappa^{s+1} = \{\eta | \eta \in \Phi_\kappa^s,\quad \pi^s\ \eta \leq\ \pi^s\gamma_\kappa^s\} \tag{19}$$

Ω_κ^{s+1} is built from Ω^s_κ by using the Malinvaud technique described above :

$$\Omega_\kappa^{s+1} = \{\eta | \eta = r\zeta + (1 - r)\gamma_\kappa^s, \qquad r \in [0,1],\ \zeta \in \Omega_\kappa^s\} \tag{20}$$

Note that Φ_κ^{s+1} is built from Φ_κ^s by eliminating irrelevant points whereas Ω_κ^{s+1} is built by adding new interesting points to Ω_κ^s.

C. Convergence

We now want to prove that $(q_1^s, \ldots, q_m^s, \gamma_1^s, \ldots, \gamma_\mu^s)$ converges to a solution of problem P2.

We assume in the same manner as above that $\{q_k^s\}$ has a limit point. For all κ, we know that $\{\zeta_\kappa^s\}$, $\{\psi_\kappa^s\}$, $\{\gamma_\kappa^s\}$ and π^s will have one as they belong to compact sets.

Consider therefore the following $m + 3\mu + 1$ sequences: $\{q_k^s\}$ $k=1, \ldots, m$; $\{\gamma_\kappa^s\}$, $\{q_\kappa^s\}$, $\{\zeta_\kappa^s\}$ $\kappa=1, \ldots, \mu$; $\{\pi^s\}$. We know that they all have a limit point. Therefore we know that there must a subset of indices, denoted S, such that the subsequences of $\{q_k^s\}$, $\{\gamma_\kappa^s\}$, $\{\psi_\kappa^s\}$, $\{\zeta_\kappa^s\}$ and $\{\pi^s\}$ corresponding to S converge simultaneously to points we will name $\bar{q}_k$, $\bar{\gamma}_\kappa$, $\bar{\psi}_\kappa$, $\bar{\zeta}_\kappa$, $\bar{\pi}$ respectively.

By (15) and (20) $\gamma_\kappa^r \in \Omega_\kappa^s$ for all $s > r$. Therefore, by (16):

$$\pi^s \gamma_\kappa^r \leq \pi^s \psi_\kappa^s \qquad \text{for all } s>r \text{ and all } \kappa. \quad (21)$$

By passing to the limit first for $s \to \infty$, then for $r \to \infty$ $(s, r) \in S$:

$$\bar{\pi}\, \gamma_\kappa^r \leq \bar{\pi}\, \bar{\psi}_\kappa \qquad (22)$$

$$\bar{\pi}\, \bar{\gamma}_\kappa \leq \bar{\pi}\, \bar{\psi}_\kappa \qquad \text{for all } \kappa$$

If we define $\bar{\gamma} = \sum_{\kappa=1}^{\mu} \bar{\gamma}_\kappa + \omega$ and $\bar{\psi} = \sum_{\kappa=1}^{\mu} \bar{\psi}_\kappa + \omega$ we have:

$$\overline{\pi}\,\overline{\gamma} \leq \overline{\pi}\,\overline{\psi} \tag{24}$$

Define $q^s = \sum_{k=1}^{m} q_k^s$, $\overline{q}_k = \sum_{k=1}^{m} \overline{q}_k$, $\zeta^s = \sum_{\kappa=1}^{\mu} \zeta_\kappa^s + \omega$,

$\overline{\zeta} = \sum_{\kappa=1}^{\mu} \overline{\zeta}_\kappa + \omega$.

Problems R_q^ss whose solution is $(\zeta_1^s, \ldots, \zeta_\mu^s)$ and T^s, whose solution is $(\psi_1^s, \ldots, \psi_\mu^s)$ have the same objective function. The domain of T^s is a subset of the domain of R_q^ss. Therefore:

$$u(q^s + \zeta^s) \geq u(q^s + \psi^s) \tag{25}$$

By continuity of u it follows that:

$$u(\overline{q} + \overline{\zeta}) \geq u(\overline{q} + \overline{\psi}) \tag{26}$$

We now want to prove that the inequality in (26) can be replaced by an equality.

Suppose that $u(\overline{q} + \overline{\zeta})$ were strictly greater than $u(\overline{q} + \overline{\psi})$. By continuity of u there would exist a δ, strictly smaller than $\overline{\zeta}$, such that:[5]

$$u(\overline{q} + \delta) > u(\overline{q} + \overline{\psi}) \tag{27}$$

Therefore there exists an N such that $s \in S$, $s > N$ implies:

[5] This technique is the one used by Malinvaud in [1].

$$u(q^s + \delta) \geq u(q^s + \psi^s) \tag{28}$$

By convexity of u and the definition of π^s:

$$\pi^s(q^s + \delta) \geq \pi^s(q^s + \psi^s) \tag{29}$$

and passing to the limit:

$$\bar{\pi}\,\delta \geq \bar{\pi}\,\bar{\psi} \tag{30}$$

as δ is strictly smaller than $\bar{\zeta}$ we have:

$$\bar{\pi}\,\bar{\zeta} > \bar{\pi}\,\bar{\psi} \tag{31}$$

(31) together with inequality (23) implies:

$$\bar{\pi}\,\bar{\zeta} > \bar{\pi}\,\bar{\gamma} \tag{32}$$

which can be rewritten:

$$\sum_{\kappa=1}^{\mu} \bar{\pi}\,\bar{\zeta}_\kappa > \sum_{\kappa=1}^{\mu} \bar{\pi}\,\bar{\gamma}_\kappa \tag{33}$$

Therefore there exists an index λ such that:

$$\bar{\pi}\,\bar{\zeta}_\lambda > \bar{\pi}\,\bar{\gamma}_\lambda \tag{34}$$

and therefore for some t, large enough:

$$\pi^t \, \bar{\zeta}_\lambda > \pi^t \, \gamma_\lambda^t \tag{35}$$

With (19), (35) would imply that $\bar{\zeta}_\lambda$ does not belong to Φ_κ^{t+1}. As ζ^{t+p} belongs to the closed set Φ_κ^{t+1} for all $p > 0$ this is impossible and (26) cannot be a strict inequality. Therefore:

$$u(\bar{q} + \bar{\zeta}) = u(\bar{q} + \bar{\psi}) \tag{36}$$

It is now easy to show that $(\bar{q}_1, \ldots, \bar{q}_m, \bar{\psi}_1, \ldots, \bar{\psi}_\mu)$ is a solution of P2. First note it is feasible. Then assume that $u(\tilde{q} + \tilde{\phi}) > u(\bar{q} + \bar{\psi})$ for some $(\tilde{q}_1, \ldots \tilde{q}_m, \tilde{\phi}_1, \ldots, \tilde{\phi}_\mu)$ feasible; $\tilde{\phi} = \sum_{\kappa=1}^{\psi} \tilde{\phi}_\kappa + \omega$. By (36), for some $s \in S$ large enough we would have:

$$u(\tilde{q} + \tilde{\phi}) > u(q^s + \zeta^s) \tag{37}$$

But (37) is not possible as $u(q^s + \zeta^s)$ is the maximum of u over a set which includes all feasible points, in particular $(\tilde{q}_1, \ldots, \tilde{q}_m, \tilde{\phi}_1, \ldots, \tilde{\phi}_\mu)$.

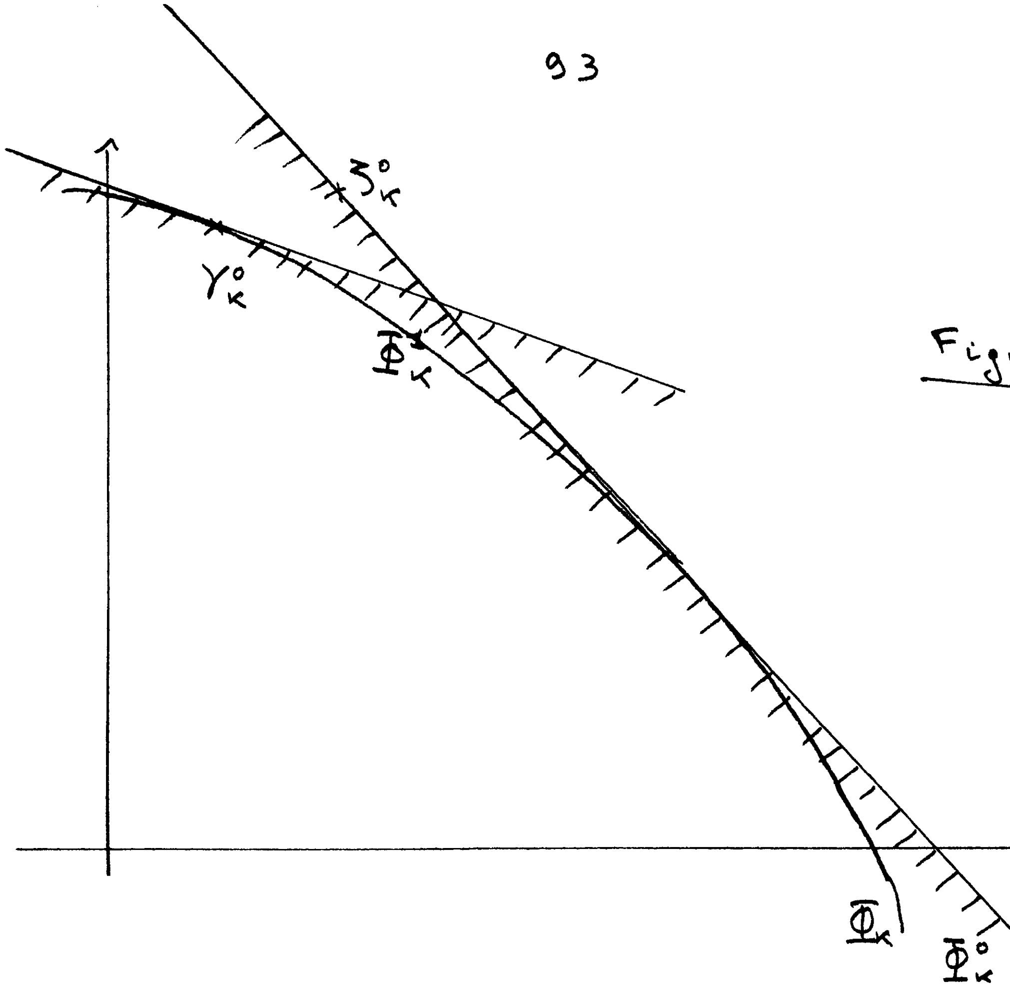


Figure 3.1

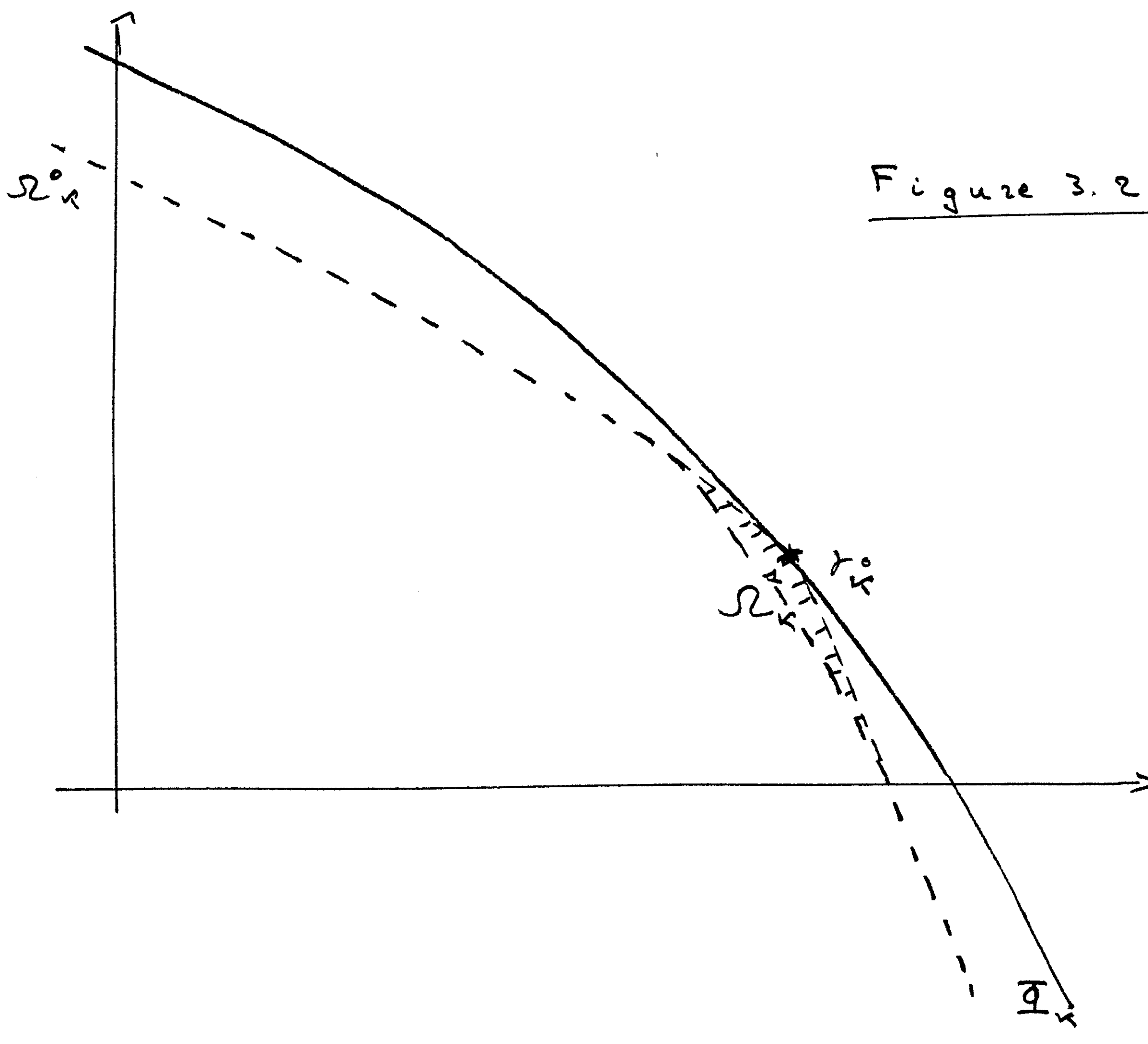
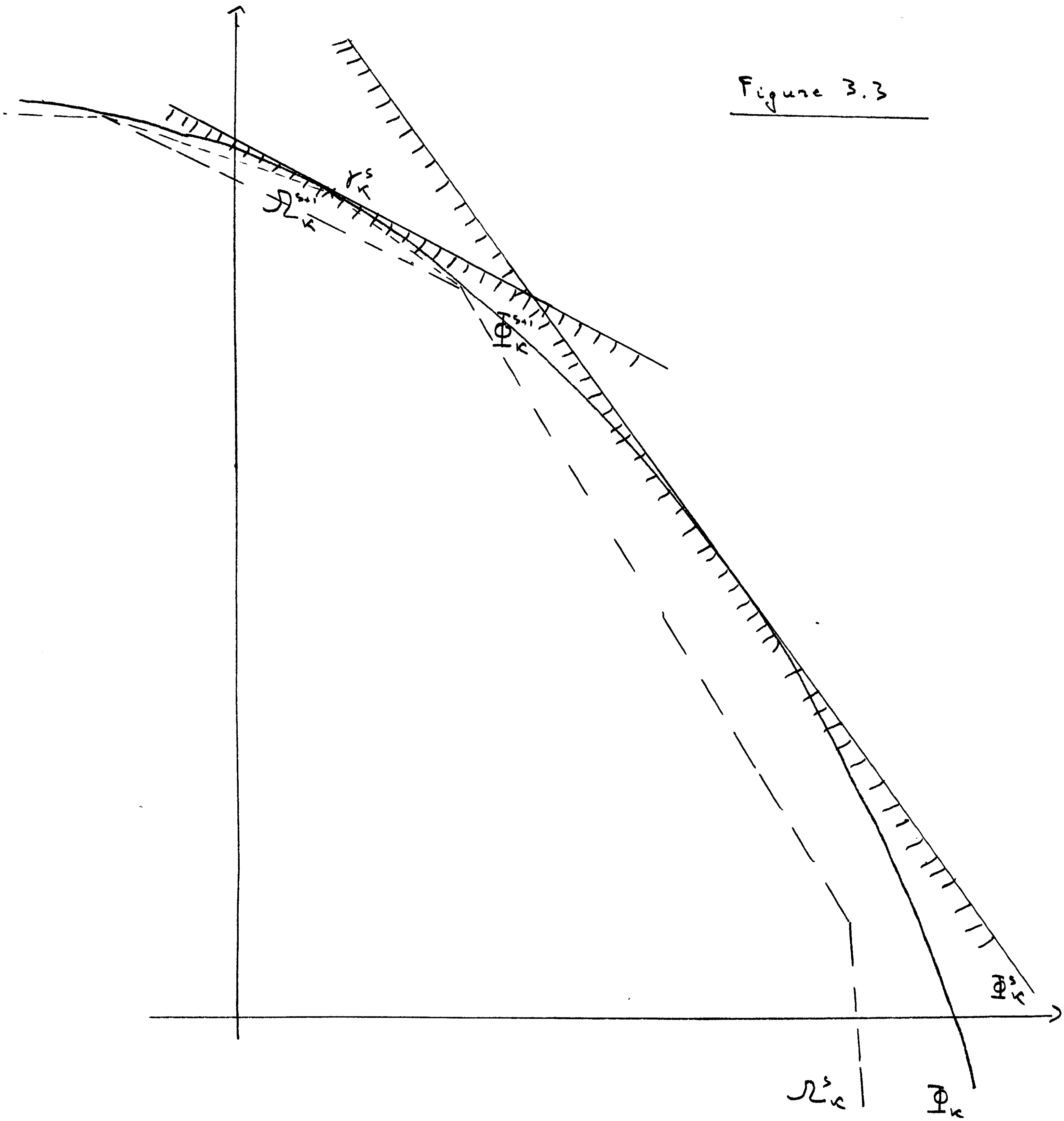


Figure 3.2

Figure 3.3



[1] Malinvaud, E. (1967), "Decentralized Procedures for Planning" in Bacharah and Malinvaud (eds.). Activity Analysis in the Theory of Growth and Planning, MacMillan, London, 170-208.

[2] Weitzman, M.L., Iterative Multilevel Planning with Production Targets. Econometrica, Vol. 38, 1970, pp 50-65.